

A functional model, eigenvalues, and finite singular critical points for indefinite Sturm-Liouville operators.

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Abstract

Eigenvalues in the essential spectrum of a weighted Sturm-Liouville operator are studied under the assumption that the weight function has one turning point. An abstract approach to the problem is given via a functional model for indefinite Sturm-Liouville operators. Algebraic multiplicities of eigenvalues are obtained. Also, operators with finite singular critical points are considered.

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1 Introduction

Let J be a signature operator in a complex Hilbert space H (i.e., $J = J^* = J^{-1}$). Then $J = P_+ - P_-$ and $H = H_+ \oplus H_-$, where P_{\pm} are the orthogonal projections onto $H_{\pm} := \ker(J \mp I)$.

Recall that a closed symmetric operator S (in a Hilbert space H) is said to be simple if there is no nontrivial reducing subspace in which S is self-adjoint.

This paper is concerned mainly with J -self-adjoint operators T such that $T_{\min} := T \cap T^*$ is a simple densely defined symmetric operator in H with the deficiency indices $n_+(T_{\min}) = n_-(T_{\min}) = 2$. This class includes weighted Sturm-Liouville operators

$$A = \frac{1}{r} \left(-\frac{d}{dx} p \frac{d}{dx} + q \right), \quad r, \frac{1}{p}, q \in L^1_{\text{loc}}(a, b), \quad -\infty \leq a < 0 < b \leq +\infty, \quad (1.1)$$

$$xr(x) > 0 \text{ a.e. on } \mathbb{R}, \quad p > 0 \text{ a.e. on } \mathbb{R}, \quad q \text{ is real-valued}, \quad (1.2)$$

equipped with separated self-adjoint boundary conditions at a and b . This statement is a consequence of the fact that the weight function r has one turning point (i.e., the point where r changes sign), see e.g. [46] and Section 2.3. (1.2) fixes the turning point of r at 0, and therefore A is J -self-adjoint in the weighted space $L^2((a, b), |r(x)|dx)$ with the operator J defined by $(Jf)(x) := (\text{sgn } x)f(x)$. Note that the case of one turning point of r is principal for applications in kinetic theory (see [6, 5, 32] and a short review in [40, Section 1]).

The eigenvalue problem for a regular indefinite Sturm-Liouville operator was studied in a number of papers starting from Hilbert [33] (see e.g. [60, 5, 4, 2, 16, 24, 66] and references therein). Till 2005, the spectral properties of singular differential operators with an indefinite weight were studied mainly under the assumption of quasi J -nonnegativity, for A this means that $\sigma(JA) \cap (-\infty, 0)$ is finite, for the definition and basic results see [16, 24]. In last decades, big attention have been attracted by the problem of similarity of A to a self-adjoint operator and the close problem of regularity of critical points (see a short review in [44]).

In this paper, the problem under consideration is a detailed description of the spectrum $\sigma(T)$ of the operator T , of the set of eigenvalues (the point spectrum) $\sigma_p(T)$, and of algebraic and geometric multiplicities of eigenvalues. In Section 2.2, after some analysis of the more general case $n_+(T_{\min}) = n_-(T_{\min}) < \infty$, we assume the above conditions on T_{\min} and construct a functional model of T based on that of symmetric operators [22, 29, 57]. It occurs that, for the operator A , the main objects of this model are the spectral measures $d\Sigma_+$ and $d\Sigma_-$ of the classical Titchmarsh-Weyl m -coefficients associated with A on $(a, 0)$ and $(0, b)$ (see Section 2.3 for details). In the abstract case, $d\Sigma_{\pm}$ are the spectral measures of two abstract Weyl functions M_{\pm} (see [21, 22] for basic facts) naturally associated with T and the signature operator J .

In Section 3, the model is used to find all eigenvalues of T and their algebraic multiplicities in terms of M_{\pm} and $d\Sigma_{\pm}$ (all geometric multiplicities are equal 1, the latter is obvious for the operator A). In turn, we obtain a description of the discrete and essential spectra and of the exceptional case when the resolvent set $\rho(T)$ is empty. For the operator A , these abstract results on the spectra of T reduce the eigenvalue problem to the problem of description of M_{\pm} and $d\Sigma_{\pm}$ (or some of their properties) in terms of coefficients p, q, r . The latter problem is difficult, but, fortunately, for some classes of coefficients is important for mathematical physics and is studied enough to get results on spectral properties of A (see Sections 3.3 and 4).

Non-emptiness of $\rho(A)$ is nontrivial and essential for the spectral analysis of A (see [4, 61] and [42, Problem 3.3]). In Section 3.3, the author generalizes slightly non-emptiness results noticed in [39, 46, 42].

A part of this paper was obtained in the author's candidate thesis [38], announced in the short communication [39], and used essentially in [46, 42]. Some of these applications, as well as connections with [44, 16, 47] and with the similarity problem, are discussed in Section 4.

Section 5 provides an alternative approach to the examples of J -self-adjoint Sturm-Liouville operators with the singular critical point 0 given in [42, Sections 5 and 6] and [44, Section 5]. A class of operators with the singular critical point 0 is constructed. Relationships of the paper [16] with the example of [42, Sections 6.1] and with Theorem 3.1 are discussed in Section 6.

The main advance of the method of the present paper is that it provides description of real eigenvalues and their algebraic multiplicities. The answer is especially nontrivial and has a rich structure in the case of embedded eigenvalues. The interest to the latter problem is partially motivated by the theory of kinetic equations of the Fokker-Plank (Kolmogorov) type (see references in Section 4.2). Also we drop completely the conditions of quasi- J -positivity and definitizability.

The method of the paper is essentially based on the abstract approach to the theory of extensions of symmetric operators via boundary triplets, e.g. [49, 21, 20]. Some results on eigenvalues of non-self-adjoint extensions of symmetric operators were obtained in [21, 22, 18, 19] with the use of this abstract approach. Relationships of these results with the results of the present paper are indicated in Remarks 2.5 and 3.4. There is a kindred approach to eigenvalue problem through characteristic functions, we refer the reader to the references in [23]. The characteristic function for the operator

A was calculated in [46, Proposition 3.9], but the analysis of [46] shows that it is difficult to apply this method to eigenvalue problem for the operator A . Connections with definitizability and local definitizability of A and T (see e.g. [54, 35] for basic facts and definitions) are given in Remarks 3.9 and 3.12.

Notation. Let H and \mathcal{H} be Hilbert spaces with the scalar products $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_{\mathcal{H}}$, respectively. The domain, kernel, and range of a (linear) operator S in H is denoted by $\text{dom}(S)$, $\ker(S)$, and $\text{ran}(S)$, respectively. If \mathfrak{D} is a subset of H , then $S\mathfrak{D}$ is the image of \mathfrak{D} , $S\mathfrak{D} := \{Sh : h \in \mathfrak{D}\}$, and $\overline{\mathfrak{D}}$ is the closure of \mathfrak{D} .

The discrete spectrum $\sigma_{\text{disc}}(S)$ of S is the set of isolated eigenvalues of finite algebraic multiplicity. The essential spectrum is defined by $\sigma_{\text{ess}}(S) := \sigma(S) \setminus \sigma_{\text{disc}}(S)$. The continuous spectrum is understood in the sense

$$\sigma_c(S) := \{\lambda \in \mathbb{C} \setminus \sigma_p(S) : \text{ran}(S - \lambda) \neq \overline{\text{ran}(S - \lambda)} = H\};$$

$\mathcal{R}_S(\lambda) := (S - \lambda I)^{-1}$, $\lambda \in \rho(S)$, is the resolvent of S . Recall that an eigenvalue λ of S is called *semi-simple* if $\ker(S - \lambda)^2 = \ker(S - \lambda)$, and *simple* if it is semi-simple and $\dim \ker(S - \lambda) = 1$. By $\mathcal{S}_\lambda(S)$ we denote the *root subspace* (the algebraic eigensubspace) of S corresponding to the point λ . That is, $\mathcal{S}_\lambda(S)$ is the closed linear hull of the subspaces $\ker(S - \lambda)^k$, $k \in \mathbb{N}$.

If S is a symmetric operator, $n_\pm(S)$ denote the deficiency indices of S (see the Appendix).

The topological support $\text{supp } d\Sigma$ of a Borel measure $d\Sigma$ on \mathbb{R} is the smallest closed set \mathcal{S} such that $d\Sigma(\mathbb{R} \setminus \mathcal{S}) = 0$; $d\Sigma(\{\lambda\})$ denotes the measure of point λ (i.e., $d\Sigma(\{\lambda\}) := \Sigma(\lambda + 0) - \Sigma(\lambda - 0)$) if the measure $d\Sigma$ is determined by a function of bounded variation Σ . We denote the indicator function of a set \mathcal{S} by $\chi_{\mathcal{S}}(\cdot)$. We write $f \in L^1_{\text{loc}}(a, b)$ ($f \in AC_{\text{loc}}(a, b)$) if the function f is Lebesgue integrable (absolutely continuous) on every closed bounded interval $[a', b'] \subset (a, b)$.

2 The functional model for indefinite Sturm-Liouville operators with one turning point

2.1 Preliminaries; the functional model of a symmetric operator

Recall a functional model of symmetric operator following [22, Section 5.2], [57, Section 7] (a close version of a functional model can be found in [29]). In this paper, we need only the case of deficiency indices $(1, 1)$.

Let $\Sigma(t)$ be a nondecreasing scalar function satisfying the conditions

$$\int_{\mathbb{R}} \frac{1}{1+t^2} d\Sigma(t) < \infty, \quad \int_{\mathbb{R}} d\Sigma(t) = \infty, \quad \Sigma(t) = \frac{1}{2}(\Sigma(t-0) + \Sigma(t+0)), \quad \Sigma(0) = 0. \quad (2.1)$$

The operator of multiplication $Q_\Sigma : f(t) \rightarrow tf(t)$ is self-adjoint in $L^2(\mathbb{R}, d\Sigma(t))$. It is assumed that Q_Σ is defined on its natural domain

$$\text{dom}(Q_\Sigma) = \{f \in L^2(\mathbb{R}, d\Sigma(t)) : \int_{\mathbb{R}} |tf(t)|^2 d\Sigma(t) < \infty\}.$$

Consider the following restriction of Q_Σ :

$$T_\Sigma = Q_\Sigma \upharpoonright \text{dom}(T_\Sigma), \quad \text{dom}(T_\Sigma) = \{f \in \text{dom}(Q_\Sigma) : \int_{\mathbb{R}} f(t) d\Sigma(t) = 0\}.$$

Then T_Σ is a simple densely defined symmetric operator in $L^2(\mathbb{R}, d\Sigma(t))$ with deficiency indices $(1,1)$. The adjoint operator T_Σ^* has the form

$$\begin{aligned} \text{dom}(T_\Sigma^*) &= \{f = f_Q + \mathfrak{c} \frac{t}{t^2 + 1} : f_Q \in \text{dom}(Q_\Sigma), \mathfrak{c} \in \mathbb{C}\}, \\ T_\Sigma^* f &= t f_Q - \mathfrak{c} \frac{1}{t^2 + 1}, \end{aligned} \quad (2.2)$$

where the constant \mathfrak{c} is uniquely determined by the inclusion $f - \mathfrak{c}t(t^2 + 1)^{-1} \in \text{dom}(Q_\Sigma)$ due to the second condition in (2.1).

Let C be a fixed real number. Define linear mappings $\Gamma_0^\Sigma, \Gamma_1^{\Sigma, C}$ from $\text{dom}(T_\Sigma^*)$ onto \mathbb{C} by

$$\Gamma_0^\Sigma f = \mathfrak{c}, \quad \Gamma_1^{\Sigma, C} f = \mathfrak{c}C + \int_{\mathbb{R}} f_Q(t) d\Sigma(t), \quad (2.3)$$

$$\text{where} \quad f = f_Q + \mathfrak{c} \frac{t}{t^2 + 1} \in \text{dom}(T_\Sigma^*), \quad f_Q \in \text{dom}(Q_\Sigma), \quad \mathfrak{c} \in \mathbb{C}.$$

Then $\{\mathbb{C}, \Gamma_0^\Sigma, \Gamma_1^{\Sigma, C}\}$ is a boundary triplet for T_Σ^* (see [22, Proposition 5.2 (3)], basic facts on boundary triplets and abstract Weyl functions are given in the Appendix). The function

$$M_{\Sigma, C}(\lambda) := C + \int_{\mathbb{R}} \left(\frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\Sigma(t), \quad \lambda \in \mathbb{C} \setminus \text{supp } d\Sigma, \quad (2.4)$$

is the corresponding Weyl function of T_Σ .

Another way to describe the operator T_Σ^* is the following (see [22]). Note that the domain $\text{dom}(T_\Sigma^*)$ consists of the functions $f \in L^2(\mathbb{R}, d\Sigma(t))$ such that for some constant $\mathfrak{c} \in \mathbb{C}$ the function $\tilde{f}(t) := tf(t) - \mathfrak{c}$ belongs to $L^2(\mathbb{R}, d\Sigma(t))$. It follows from (2.1) that the constant \mathfrak{c} is uniquely determined and coincides with the constant \mathfrak{c} introduced in (2.2). Therefore,

$$\mathfrak{c} = \Gamma_0^\Sigma f \quad \text{and} \quad T_\Sigma^* f = \tilde{f}. \quad (2.5)$$

2.2 The functional model for J-self-adjoint extensions of symmetric operators

Let J be a signature operator in a Hilbert space H , i.e., $J = J^* = J^{-1}$. Then $J = P_+ - P_-$ and $H = H_+ \oplus H_-$, where P_\pm are the orthogonal projections onto $H_\pm := \ker(J \mp I)$.

Let T be a J -self-adjoint operator in H , i.e., the operator $B = JT$ is self-adjoint. The domains of T and B coincide, we denote them by

$$\mathcal{D} := \text{dom}(T) (= \text{dom}(B)).$$

Put

$$T_{\min} := T \cap T^*, \quad \mathcal{D}_{\min} := \text{dom}(T_{\min}).$$

By the definition, the operator T_{\min} is a symmetric operator and so is

$$B_{\min} := B \upharpoonright \mathcal{D}_{\min} = JT_{\min}. \quad (2.6)$$

Let Σ_+ and Σ_- be nondecreasing scalar functions satisfying (2.1). Let C_+ and C_- be real constants. Consider the operator $\widehat{A} = \widehat{A} \{\Sigma_+, C_+, \Sigma_-, C_-\}$ in $L^2(\mathbb{R}, d\Sigma_+) \oplus L^2(\mathbb{R}, d\Sigma_-)$ defined by

$$\widehat{A} \{\Sigma_+, C_+, \Sigma_-, C_-\} = T_{\Sigma_+}^* \oplus T_{\Sigma_-}^* \upharpoonright \text{dom}(\widehat{A}), \quad (2.7)$$

$$\text{dom}(\widehat{A}) = \{ f = f_+ + f_- : f_{\pm} \in \text{dom}(T_{\Sigma_{\pm}}^*),$$

$$\Gamma_0^{\Sigma_+} f_+ = \Gamma_0^{\Sigma_-} f_-, \Gamma_1^{\Sigma_+, C_+} f_+ = \Gamma_1^{\Sigma_-, C_-} f_- \},$$

where $T_{\Sigma_{\pm}}^*$ are the operators defined in Subsection 2.1.

One of main results of this paper is the following theorem.

Theorem 2.1. *Let J be a signature operator in a separable Hilbert space H and let T be a J -self-adjoint operator such that $T_{\min} := T \cap T^*$ is a simple densely defined symmetric operator in H with deficiency indices $(2, 2)$. Then there exist nondecreasing scalar functions Σ_+, Σ_- satisfying (2.1) and real constants C_+, C_- such that T is unitarily equivalent to the operator $\widehat{A} \{\Sigma_+, C_+, \Sigma_-, C_-\}$.*

First, we prove several propositions that describe the structure of T as an extension of the symmetric operator T_{\min} , and then prove Theorem 2.1 at the end of this subsection.

Proposition 2.2. *Let T be a J -self-adjoint operator. Let the operators T_{\min}^{\pm} be defined by*

$$T_{\min}^{\pm} := T \upharpoonright \mathcal{D}_{\min}^{\pm}, \quad \mathcal{D}_{\min}^{\pm} = \text{dom}(T_{\min}^{\pm}) := \mathcal{D}_{\min} \cap H_{\pm}. \quad (2.8)$$

Then:

(i) T_{\min}^{\pm} is a symmetric operator in the Hilbert space H_{\pm} and

$$T_{\min} = T_{\min}^+ \oplus T_{\min}^-, \quad B_{\min} = B_{\min}^+ \oplus B_{\min}^-, \quad \text{where} \quad B_{\min}^{\pm} := \pm T_{\min}^{\pm}. \quad (2.9)$$

(ii) If any of the following two conditions

$$(a) \quad \rho(T) \neq \emptyset,$$

$$(b) \quad n_+(T_{\min}) = n_-(T_{\min}),$$

$$\text{is satisfied, then } n_+(T_{\min}^+) = n_-(T_{\min}^+) \text{ and } n_+(T_{\min}^-) = n_-(T_{\min}^-).$$

In particular, (a) implies (b).

Proof. (i) Since $B = B^*$ and $\mathcal{D} = \text{dom}(T) = \text{dom}(B)$, we have $T^* = BJ$ and $\mathcal{D}_{\min} = \{f \in \mathcal{D} \cap J\mathcal{D} : JBf = BJf\}$. So if $f \in \mathcal{D}_{\min}$ and $g = Jf$, then $g \in \mathcal{D} \cap J\mathcal{D}$ and

$$JBg = JBJf = JJBf = Bf = BJg.$$

This implies $J\mathcal{D}_{\min} \subset \mathcal{D}_{\min}$ (and in turn $J\mathcal{D}_{\min} = \mathcal{D}_{\min}$ since J is a unitary operator). Hence, for $f \in \mathcal{D}_{\min}$ we have $P_+f + P_-f \in \mathcal{D}_{\min}$ and $P_+f - P_-f \in \mathcal{D}_{\min}$. So $P_{\pm}f \in \mathcal{D}_{\min}$ and $\mathcal{D}_{\min} \subset (\mathcal{D}_{\min} \cap H_+) \oplus (\mathcal{D}_{\min} \cap H_-)$. The inverse inclusion is obvious, and we see that

$$\mathcal{D}_{\min} = (\mathcal{D}_{\min} \cap H_+) \oplus (\mathcal{D}_{\min} \cap H_-).$$

Now note that $T_{\min}(\mathcal{D}_{\min} \cap H_{\pm}) \subset H_{\pm}$. Indeed, let $f_{\pm} \in \mathcal{D}_{\min} \cap H_{\pm}$. Since $Jf_{\pm} = \pm f_{\pm}$ and $Tf_{\pm} = T^*f_{\pm}$, we see that

$$JBf_{\pm} = BJf_{\pm} = \pm Bf_{\pm}. \quad (2.10)$$

Note that $g \in H_{\pm}$ is equivalent to $Jg = \pm g$. So (2.10) implies $Bf_{\pm} \in H_{\pm}$, and therefore the vector $T_{\min}f_{\pm} = Tf_{\pm} = JBf_{\pm}$ belongs to H_{\pm} .

The first part of (2.9) is proved. Since T_{\min} is a symmetric operator in H , the operators T_{\min}^{\pm} are symmetric too. Finally, the second part of (2.9) follows from (2.6) and (2.10).

(ii) Since $B = B^*$, it easy to see that

$$n_+(B_{\min}^+) + n_+(B_{\min}^-) = n_+(B_{\min}) = n_-(B_{\min}) = n_-(B_{\min}^+) + n_-(B_{\min}^-) =: m. \quad (2.11)$$

The equalities $n_{\pm}(T_{\min}^+) = n_{\pm}(B_{\min}^+)$ and $n_{\pm}(T_{\min}^-) = n_{\mp}(B_{\min}^-)$ imply

$$n_{\pm}(T_{\min}) = n_{\pm}(T_{\min}^+) + n_{\pm}(T_{\min}^-) = n_{\pm}(B_{\min}^+) + n_{\mp}(B_{\min}^-). \quad (2.12)$$

It follows from (2.11) and (2.12) that $n_+(T_{\min}) > m$ yields $n_-(T_{\min}) < m$. In this case, $\mathbb{C}_- \subset \sigma_p(T)$ and $H \neq \overline{(T - \lambda I) \operatorname{dom}(T)}$ for $\lambda \in \mathbb{C}_+$. Hence, $\rho(T) = \emptyset$. The case $n_+(T_{\min}) < m$, $n_-(T_{\min}) > m$ is similar. Thus, if $\rho(T) \neq \emptyset$ or $n_+(T_{\min}) = n_-(T_{\min})$, then

$$n_+(B_{\min}^+) + n_-(B_{\min}^-) = n_-(B_{\min}^+) + n_+(B_{\min}^-) = m.$$

Using (2.11), we get $n_+(B_{\min}^{\pm}) = n_-(B_{\min}^{\pm})$ and, therefore, $n_+(T_{\min}^{\pm}) = n_-(T_{\min}^{\pm})$. \square

Assume now that the operator T_{\min} is densely defined in H . Put, for convenience' sake,

$$T_{\max}^{\pm} = (T_{\min}^{\pm})^*, \quad B_{\max}^{\pm} = (B_{\min}^{\pm})^*.$$

Clearly,

$$T_{\max}^{\pm} = \pm B_{\max}^{\pm} \quad \text{and} \quad \operatorname{dom}(T_{\max}^{\pm}) = \operatorname{dom}(B_{\max}^{\pm}) =: \mathcal{D}_{\max}^{\pm}. \quad (2.13)$$

Proposition 2.3. *Let T be a J -self-adjoint operator. Assume that T_{\min} is densely defined in H and*

$$n_+(T_{\min}^+) = n_-(T_{\min}^+) =: N^+ < \infty, \quad n_+(T_{\min}^-) = n_-(T_{\min}^-) =: N^- < \infty.$$

Then:

- (i) $n_+(T_{\min}^+) = n_-(T_{\min}^+) = n_+(T_{\min}^-) = n_-(T_{\min}^-)$, that is, $N^+ = N^- =: N$;
- (ii) the mappings $\mathfrak{P}_{\pm} := P_{\pm} \upharpoonright \mathcal{D}/\mathcal{D}_{\min}$ are well-defined and are linear isomorphisms from the quotient space $\mathcal{D}/\mathcal{D}_{\min}$ onto the quotient space $\mathcal{D}_{\max}^{\pm}/\mathcal{D}_{\min}^{\pm}$.

Proof. Note that

$$\mathcal{D}_{\max} := \mathcal{D}_{\max}^+ \oplus \mathcal{D}_{\max}^- \quad (2.14)$$

is a domain of both the operators $T_{\max} := T_{\min}^*$ and $B_{\max} := B_{\min}^*$. Since

$$\mathcal{D} = \operatorname{dom}(B) \subset \operatorname{dom}(B_{\max}) \quad \text{and} \quad P_{\pm} \mathcal{D}_{\min} = \mathcal{D}_{\min}^{\pm},$$

we see that $\mathfrak{P}_{\pm} : \mathcal{D}/\mathcal{D}_{\min} \rightarrow \mathcal{D}_{\max}^{\pm}/\mathcal{D}_{\min}^{\pm}$ are well-defined linear mappings.

Let us show that

$$\text{the mappings } \mathfrak{P}_\pm \text{ are injective.} \quad (2.15)$$

Indeed, if $\ker \mathfrak{P}_+ \neq \{0\}$, then there exists $h \in \mathcal{D}$ such that $h \notin \mathcal{D}_{\min}$ and $P_+h \in \mathcal{D}_{\min}^+$. Recall that $\mathcal{D}_{\min}^+ \subset \mathcal{D}$, so $P_-h = h - P_+h \in \mathcal{D} \cap H_- = \mathcal{D}_{\min}^-$. By the first equality in (2.9), $\mathcal{D}_{\min} = \mathcal{D}_{\min}^+ \oplus \mathcal{D}_{\min}^-$ and this implies that $h = P_+h + P_-h$ belongs to \mathcal{D}_{\min} , a contradiction.

Since $2N^\pm = \dim(\mathcal{D}_{\max}^\pm / \mathcal{D}_{\min}^\pm)$, it follows from (2.15) that

$$2N^+ \geq m, \quad 2N^- \geq m, \quad \text{where } m := \dim(\mathcal{D} / \mathcal{D}_{\min}) \quad (2.16)$$

(this definition of m coincides with that of (2.11)). Indeed, B is a self-adjoint extension of B_{\min} , therefore,

$$\dim(\mathcal{D} / \mathcal{D}_{\min}) = n_+(B_{\min}^+) + n_+(B_{\min}^-) = n_-(B_{\min}^+) + n_-(B_{\min}^-).$$

We see that $m = N^+ + N^-$. From this and (2.16), we get $N^+ = N^- = m/2$. Thus, statement (i) holds true. Besides, taking (2.15) and $N^\pm < \infty$ into account, one obtains that \mathfrak{P}_\pm are surjective. This completes the proof of (ii). \square

Recall that existence of a boundary triplet for S^* , where S is a symmetric operator in a separable Hilbert space H , is equivalent to $n_+(S) = n_-(S)$ (see [49, 21]).

Theorem 2.4 (cf. Theorem 6.4 of [20]). *Let T be a J -self-adjoint operator. Assume that T_{\min} is densely defined in H and $n_+(T_{\min}) = n_-(T_{\min}) =: m < \infty$. Then:*

(i) *m is an even number and*

$$n_+(T_{\min}^+) = n_-(T_{\min}^+) = n_+(T_{\min}^-) = n_-(T_{\min}^-) = m/2.$$

(ii) *Let $\{\mathbb{C}^{m/2}, \Gamma_0^+, \Gamma_1^+\}$ be a boundary triple for T_{\max}^+ . Then there exist a boundary triple $\{\mathbb{C}^{m/2}, \Gamma_0^-, \Gamma_1^-\}$ for T_{\max}^- such that*

$$\mathcal{D} = \{h \in \mathcal{D}_{\max} : \Gamma_0^+ P_+ h = \Gamma_0^- P_- h, \quad \Gamma_1^+ P_+ h = \Gamma_1^- P_- h\} \quad (2.17)$$

(note that $P_\pm h \in \mathcal{D}_{\max}^\pm$ due to (2.14)).

Theorem 2.4 shows that the operator T admits the representation

$$T = T_{\max}^+ \oplus T_{\max}^- \upharpoonright \mathcal{D}, \quad \text{and its domain } \mathcal{D} \text{ has the representation (2.17).} \quad (2.18)$$

Proof. (i) follows from Proposition 2.3 (i).

(ii) Let $\{\mathbb{C}^{m/2}, \Gamma_0^+, \Gamma_1^+\}$ be a boundary triple for the operator T_{\max}^+ (actually, statement (i) implies that such a boundary triple exists, for the case when the space H is separable see e.g. [49]). It follows from Definition A.1, that $\Gamma_0^+ \mathcal{D}_{\min}^+ = \Gamma_1^+ \mathcal{D}_{\min}^+ = \{0\}$. So one can consider the mappings $\Gamma^+ : h_+ \rightarrow \{\Gamma_0^+ h_+, \Gamma_1^+ h_+\}$ as a linear isomorphism from $\mathcal{D}_{\max}^+ / \mathcal{D}_{\min}^+$ onto $\mathbb{C}^{m/2} \oplus \mathbb{C}^{m/2}$. Introducing the mappings

$$\Gamma_j^- := \Gamma_j^+ \mathfrak{P}_+ \mathfrak{P}_+^{-1}, \quad j = 0, 1, \quad (2.19)$$

one can get from Proposition 2.3 (ii) the fact that $\Gamma^- : h_- \rightarrow \{\Gamma_0^- h_-, \Gamma_1^- h_-\}$ is a linear isomorphism from $\mathcal{D}_{\max}^- / \mathcal{D}_{\min}^-$ onto $\mathbb{C}^{m/2} \oplus \mathbb{C}^{m/2}$. Putting $\Gamma_0^- h_- = \Gamma_1^- h_- = 0$ for all $h_- \in \mathcal{D}_{\min}^-$, we get natural linear extensions of Γ_0^- , Γ_1^- , and Γ^- on \mathcal{D}_{\max}^- .

Let $\mathfrak{h} \in \mathcal{D}_{\max}/\mathcal{D}_{\min}$ and $\mathfrak{h}_{\pm} = \tilde{P}_{\pm}\mathfrak{h}$, where \tilde{P}_{\pm} are mappings from $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ to $\mathcal{D}_{\max}^{\pm}/\mathcal{D}_{\min}^{\pm}$ induced by P_{\pm} . By Proposition 2.3 (ii), $\mathfrak{h} \in \mathcal{D}/\mathcal{D}_{\min}$ if and only if $\mathfrak{h}_{+} = \mathfrak{P}_{+}\mathfrak{P}_{-}^{-1}\mathfrak{h}_{-}$. From this and (2.19), one can obtain easily that

$$\mathcal{D} = \{h = h_{+} + h_{-} : h_{\pm} \in \mathcal{D}_{\max}^{\pm}, \Gamma_j^{+}h_{+} = \Gamma_j^{-}h_{-}, j = 0, 1\}. \quad (2.20)$$

Let us show that $\{\mathbb{C}^{m/2}, \Gamma_0^{-}, \Gamma_1^{-}\}$ is a boundary triple for T_{\max}^{-} . The property (ii) of Definition A.1 follows from the same property for the boundary triple $\{\mathbb{C}^{m/2}, \Gamma_0^{+}, \Gamma_1^{+}\}$ and from Proposition 2.3 (ii).

Now we have to prove property (i) of Definition A.1. Since $B = B^{*}$, for all $f, g \in \mathcal{D} = \text{dom}(B)$ we have

$$\begin{aligned} 0 &= (Bf, g)_H - (f, Bg)_H = \\ &= (B_{\max}(P_{+}f + P_{-}f), P_{+}g + P_{-}g)_H - (P_{+}f + P_{-}f, B_{\max}(P_{+}g + P_{-}g))_H = \\ &= (B_{\max}^{+}P_{+}f, P_{+}g)_H + (B_{\max}^{-}P_{-}f, P_{-}g)_H - (P_{+}f, B_{\max}^{+}P_{+}g)_H - (P_{-}f, B_{\max}^{-}P_{-}g)_H \end{aligned} \quad (2.21)$$

Since $P_{\pm}f, P_{\pm}g \in \mathcal{D}_{\max}^{\pm}$ and $\{\mathbb{C}^{m/2}, \Gamma_0^{+}, \Gamma_1^{+}\}$ is a boundary triple for $T_{\max}^{+} = B_{\max}^{+}$, Definition A.1 yields

$$\begin{aligned} (B_{\max}^{+}P_{+}f, P_{+}g)_H - (P_{+}f, B_{\max}^{+}P_{+}g)_H &= \\ &= (\Gamma_1^{+}P_{+}f, \Gamma_0^{+}g)_{\mathbb{C}^{m/2}} - (\Gamma_0^{+}P_{+}f, \Gamma_1^{+}P_{+}g)_{\mathbb{C}^{m/2}}. \end{aligned} \quad (2.22)$$

From (2.20) and $f, g \in \mathcal{D}$, we get

$$\begin{aligned} (\Gamma_1^{+}P_{+}f, \Gamma_0^{+}P_{+}g)_{\mathbb{C}^{m/2}} - (\Gamma_0^{+}P_{+}f, \Gamma_1^{+}P_{+}g)_{\mathbb{C}^{m/2}} &= \\ &= (\Gamma_1^{-}P_{-}f, \Gamma_0^{-}P_{-}g)_{\mathbb{C}^{m/2}} - (\Gamma_0^{-}P_{-}f, \Gamma_1^{-}P_{-}g)_{\mathbb{C}^{m/2}}. \end{aligned} \quad (2.23)$$

It follows from (2.21), (2.22), and (2.23) that

$$\begin{aligned} 0 &= (\Gamma_1^{-}P_{-}f, \Gamma_0^{-}P_{-}g)_{\mathbb{C}^{m/2}} - (\Gamma_0^{-}P_{-}f, \Gamma_1^{-}P_{-}g)_{\mathbb{C}^{m/2}} + \\ &\quad + (B_{\max}^{-}P_{-}f, P_{-}g)_H - (P_{-}f, B_{\max}^{-}P_{-}g)_H. \end{aligned}$$

or, equivalently,

$$\begin{aligned} (T_{\max}^{-}P_{-}f, P_{-}g)_H - (P_{-}f, T_{\max}^{-}P_{-}g)_H &= \\ &= (\Gamma_1^{-}P_{-}f, \Gamma_0^{-}P_{-}g)_{\mathbb{C}^{m/2}} - (\Gamma_0^{-}P_{-}f, \Gamma_1^{-}P_{-}g)_{\mathbb{C}^{m/2}} \end{aligned} \quad (2.24)$$

for all $f, g \in \mathcal{D}$. It follows easily from Proposition 2.3 (ii) that the mapping $P_{-} : \mathcal{D} \rightarrow \mathcal{D}_{\max}^{-}$ is surjective. Therefore (2.24) implies that property (ii) of Definition A.1 is fulfilled for $\{\mathbb{C}^{m/2}, \Gamma_0^{-}, \Gamma_1^{-}\}$ and so this triple is a boundary triple for T_{\max}^{-} . Finally, note that (2.20) coincides with (2.17). \square

Proof of Theorem 2.1. By Theorem 2.4 (i), $n_{\pm}(T_{\min}^{\pm}) = 1$ and there exist boundary triplets $\Pi^{\pm} = \{\mathbb{C}, \Gamma_0^{\pm}, \Gamma_1^{\pm}\}$ for T_{\max}^{\pm} such that (2.17) holds. Let M_{\pm} be the Weyl functions of T_{\min}^{\pm} corresponding to the boundary triplets Π^{\pm} . Since T_{\min}^{\pm} are densely defined operators, M_{\pm} have the form (2.4) with certain constants $C_{\pm} \in \mathbb{R}_{\pm}$ and positive measures $d\Sigma_{\pm}(t)$ satisfying (2.1). This fact follows from Corollary 2 in [21, Section 1.2] as well as from the remark after [22, Theorem 1.1] and [22, Remark 5.1] (note that, in the case of deficiency indices (1,1), condition (3) of Corollary 2 in [21, Section

1.2] is equivalent to the second condition in (2.1)). By Corollary 1 in [21, Section 1.2] (see also [21, Corollary 7.1]), the simplicity of both the operators T_{\min}^{\pm} and $\widehat{T}_{\Sigma_{\pm}}$ implies that

$$U_{\pm} T_{\min}^{\pm} U_{\pm}^{-1} = T_{\Sigma_{\pm}}, \quad (2.25)$$

where $T_{\Sigma_{\pm}}$ are the operators defined in Subsection 2.1, and U_{\pm} are certain unitary operators from H_{\pm} onto $L^2(\mathbb{R}, d\Sigma_{\pm}(t))$. Moreover, the unitary operators U_{\pm} can be chosen such that

$$\Gamma_0^{\pm} = \Gamma_0^{\Sigma_{\pm}} U_{\pm}, \quad \Gamma_1^{\pm} = \Gamma_1^{\Sigma_{\pm}, C_{\pm}} U_{\pm}. \quad (2.26)$$

The last statement follows from the description of all possible boundary triples in terms of chosen one (see e.g. [49] and [22, Proposition 1.7]). Indeed, since the deficiency indices of T_{\min}^{\pm} are (1,1), [22, formulae (1.12) and (1.13)] imply that $\Gamma_0^{\pm} = e^{i\alpha_{\pm}} \Gamma_0^{\Sigma_{\pm}} U_{\pm}$ and $\Gamma_1^{\pm} = e^{i\alpha_{\pm}} \Gamma_1^{\Sigma_{\pm}, C_{\pm}} U_{\pm}$ with $\alpha \in [0, 2\pi)$. Now changing U_{\pm} to $e^{i\alpha_{\pm}} U_{\pm}$ we save (2.25) and get (2.26).

Formulae (2.18) and (2.17) complete the proof. \square

Remark 2.5. (1) Self-adjoint couplings of symmetric operators were studied in [20, 26] (see also references therein). Theorem 2.4 (ii) can be considered as a modification of [20, Theorem 6.4] for J -self-adjoint operators.

(2) Note that in Proposition 2.2 we *do not* assume that the domain \mathcal{D}_{\min} of T_{\min} is dense in H . However, for convenience' sake, the operator T_{\min} is assumed to be densely defined in the other statements of this subsection. The assumption $\overline{\text{dom}(T_{\min})} = H$ can be removed from Proposition 2.3 and Theorem 2.4 with the use of the linear relation notion in the way similar to [20, Section 6].

(3) Theorems 2.4 (ii) and 2.1 show that the operator T admits an infinite family of functional models, which corresponds to the infinite family of boundary triples. All the functional models can be derived from a chosen one due to [22, Proposition 1.7].

2.3 The Sturm-Liouville case

Consider the differential expressions

$$\mathfrak{l}[y] = \frac{1}{|r|} (-(py')' + qy) \quad \text{and} \quad \mathfrak{a}[y] = \frac{1}{r} (-(py')' + qy), \quad (2.27)$$

assuming that $1/p, q, r \in L_{loc}^1(a, b)$ are real-valued coefficients, that $\frac{1}{p(x)} > 0$ and $xr(x) > 0$ for almost all $x \in (a, b)$, and that $-\infty \leq a < 0 < b \leq +\infty$. So the weight function r has the only turning point at 0 and the differential expressions \mathfrak{a} and \mathfrak{l} are regular at all points of the interval (a, b) (but may be singular at the endpoints a and b). The differential expressions are understood in the sense of M.G. Krein's quasi-derivatives (see e.g. [16]).

If the endpoint a (the endpoint b) is regular or is in the limit circle case for $\mathfrak{l}[\cdot]$, we equip the expressions $\mathfrak{l}[\cdot]$ and $\mathfrak{a}[\cdot]$ with a separated self-adjoint boundary condition (see e.g. [65] or [54]) at a (resp., b), and get in this way the self-adjoint operator L and the J -self-adjoint operator A in the Hilbert space $L^2(\mathbb{R}, |r(x)|dx)$. Indeed, $A = JL$ with J defined by

$$(Jf)(x) = (\text{sgn } x)f(x), \quad (2.28)$$

Obviously, $J^* = J^{-1} = J$ in $L^2((a, b), |r(x)|dx)$. So J is a signature operator and A is a J -self-adjoint operator.

In the case when $\mathfrak{l}[\cdot]$ is in the limit point case at a and/or b , we do not need boundary conditions at a and/or b .

It is not difficult to see that the operator $A_{\min} := A \cap A^*$ is a closed densely defined symmetric operator with the deficiency indices (2,2) and that A_{\min} admits an orthogonal decomposition $A_{\min} = A_{\min}^+ \oplus A_{\min}^-$, where A_{\min}^+ (A_{\min}^-) is a part of A_{\min} in $L^2((0, b), |r(x)|dx)$ (resp., $L^2((a, 0), |r(x)|dx)$), see e.g. [46, Section 2.1], and (2.31) below for a particular case (note that A_{\min} is *not* a minimal operator associated with $\mathfrak{a}[\cdot]$ in the usual sense). The operators A_{\min}^{\pm} are simple. This fact considered known by specialists, it was proved in [30], formally, under some additional conditions on the coefficients. A modification of the same proof is briefly indicated in Remark 2.7 below. So A_{\min} is a simple symmetric operator.

Applying Theorem 2.1, one obtains a functional model for A . However, we will show that a model for A can be obtained directly from the classical spectral theory of Sturm-Liouville operators and that Σ_{\pm} are spectral measures associated with Titchmarsh-Weyl m -coefficients of A .

To avoid superfluous notation and consideration of several different cases, we argue for the case when

$$(a, b) = \mathbb{R}, \quad p \equiv 1, \quad r(x) \equiv \operatorname{sgn} x, \quad (2.29)$$

$$\text{and the differential expression } \mathfrak{l}[\cdot] \text{ is limit-point at } +\infty \text{ and } -\infty. \quad (2.30)$$

That is we assume that the operator

$$L = -\frac{d^2}{dx^2} + q(x) \quad \left(A = (\operatorname{sgn} x) \left(-\frac{d^2}{dx^2} + q(x) \right) \right)$$

is defined on the maximal domain and is self-adjoint (resp., J-self-adjoint). Under these assumptions,

$$\operatorname{dom}(L) = \operatorname{dom}(A) = \{y \in L^2(\mathbb{R}) : y, y' \in AC_{\text{loc}}(\mathbb{R}), y'' + qy \in L^2(\mathbb{R})\}.$$

The operator $A_{\min} = A \cap A^*$ has the form

$$\begin{aligned} A_{\min} &= A \upharpoonright \operatorname{dom}(A_{\min}), \\ \operatorname{dom}(A_{\min}) &= \{y \in \operatorname{dom}(A) : y(0) = y'(0) = 0\}. \end{aligned} \quad (2.31)$$

By A_{\min}^{\pm} we define the restrictions of A_{\min} on $\operatorname{dom}(A_{\min}) \cap L^2(\mathbb{R}_{\pm})$.

Let us define the Titchmarsh-Weyl m -coefficients $M_{N+}(\lambda)$ and $M_{N-}(\lambda)$ for the Neumann problem associated with the differential expression $\mathfrak{a}[\cdot]$ on \mathbb{R}_+ and \mathbb{R}_- , respectively. Facts mentioned below can be found, e.g., in [56, 63], where they are given for spectral problems on \mathbb{R}_+ , but the modification for \mathbb{R}_- is straightforward. Let $s(x, \lambda)$, $c(x, \lambda)$ be the solutions of the equation

$$-y''(x) + q(x)y(x) = \lambda y(x)$$

subject to boundary conditions

$$s(0, \lambda) = \frac{d}{dx}c(0, \lambda) = 0, \quad \frac{d}{dx}s(0, \lambda) = c(0, \lambda) = 1.$$

Then $M_{N\pm}(\lambda)$ are well-defined by the inclusions

$$\psi_{\pm}(\cdot, \lambda) = -s(\cdot, \pm\lambda) + M_{N\pm}(\lambda) c(\cdot, \pm\lambda) \in L^2(\mathbb{R}_{\pm}), \quad (2.32)$$

for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

The functions $M_{N\pm}(\lambda)$ are (R)-functions (belong to the class (R)) ; i.e., $M_{N\pm}(\lambda)$ are holomorphic in $\mathbb{C} \setminus \mathbb{R}$, $M_{N\pm}(\bar{\lambda}) = \overline{M_{N\pm}(\lambda)}$ and $\text{Im } \lambda \text{ Im } M_{N\pm}(\lambda) \geq 0$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (see e.g. [37]).

Moreover, $M_{N\pm}(\lambda)$ admit the following representation

$$M_{N\pm}(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_{N\pm}(t)}{t - \lambda}, \quad (2.33)$$

where $\Sigma_{N\pm}$ are nondecreasing scalar function such that conditions (2.1) are fulfilled and

$$\int_{\mathbb{R}} (1 + |t|)^{-1} d\Sigma_{N\pm}(t) < \infty;$$

the functions $M_{N\pm}(\lambda)$ have the asymptotic formula

$$M_{N\pm}(\lambda) = \pm \frac{i}{\sqrt{\pm\lambda}} + O\left(\frac{1}{\lambda}\right), \quad (\lambda \rightarrow \infty, \quad 0 < \delta < \arg \lambda < \pi - \delta). \quad (2.34)$$

Here and below \sqrt{z} is the branch of the multifunction on the complex plane \mathbb{C} with the cut along \mathbb{R}_+ , singled out by the condition $\sqrt{-1} = i$. We assume that $\sqrt{\lambda} \geq 0$ for $\lambda \in [0, +\infty)$.

Let A_0^\pm be the self-adjoint operators associated with the Neumann problem $y'(\pm 0) = 0$ for the differential expression $\mathfrak{a}[\cdot]$ on \mathbb{R}_\pm . The measures $d\Sigma_{N\pm}(t)$ are called *the spectral measures* of the operators A_0^\pm since

$$Q_{\Sigma_{N\pm}} = \mathcal{F}_\pm A_0^\pm \mathcal{F}_\pm^{-1}$$

where $Q_{\Sigma_{N\pm}}$ are the operators of multiplication by t in the space $L^2(\mathbb{R}, d\Sigma_{N\pm}(t))$ and \mathcal{F}_\pm are the (generalized) Fourier transformations defined by

$$(\mathcal{F}_\pm f)(t) := \text{l.i.m.}_{x_1 \rightarrow \pm\infty} \pm \int_0^{x_1} f(x) c(x, \pm t) dx. \quad (2.35)$$

Here l.i.m. denotes the strong limit in $L^2(\mathbb{R}, d\Sigma_{N\pm})$. Recall that \mathcal{F}_\pm are unitary operators from $L^2(\mathbb{R}_\pm)$ onto $L^2(\mathbb{R}, d\Sigma_{N\pm})$.

Note that $\text{supp } d\Sigma_{N\pm} = \sigma(Q_{\Sigma_{N\pm}}) = \sigma(A_0^\pm)$, that (2.33) gives a holomorphic continuation of $M_{N\pm}(\lambda)$ to $\mathbb{C} \setminus \text{supp } d\Sigma_{N\pm}$, and that, in this domain, $M_{N\pm}(\lambda) = M_{\Sigma_{N\pm}, C_{N\pm}}(\lambda)$, where

$$C_{N\pm} := \int_{\mathbb{R}} \frac{t}{1 + t^2} d\Sigma_{N\pm} \quad (2.36)$$

and $M_{\Sigma_{N\pm}, C_{N\pm}}(\lambda)$ are defined by (2.4).

Theorem 2.6. *Assume that conditions (2.29) and (2.30) are fulfilled and the J -self-adjoint operator $A = (\text{sgn } x)(-d^2/dx^2 + q(x))$ is defined as above. Then A is unitarily equivalent to the operator $\hat{A} = \hat{A}\{\Sigma_{N+}, C_{N+}, \Sigma_{N-}, C_{N-}\}$. More precisely,*

$$(\mathcal{F}_+ \oplus \mathcal{F}_-) A (\mathcal{F}_+^{-1} \oplus \mathcal{F}_-^{-1}) = \hat{A}. \quad (2.37)$$

Proof. The proof is based on two following representations of the resolvent $\mathcal{R}_{A_0^\pm}$ (see [56, 64]):

$$(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)(x) = \mp\psi_\pm(x, \lambda) \int_0^{\pm x} c(s, \pm\lambda)f(s)ds \mp c(x, \pm\lambda) \int_{\pm x}^{\pm\infty} \psi_\pm(s, \lambda)f(s)ds, \quad (2.38)$$

$$(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)(x) = \int_{\mathbb{R}} \frac{c(x, \pm t) (\mathcal{F}_\pm f_\pm)(t) d\Sigma_\pm(t)}{t - \lambda}, \quad x \in \mathbb{R}_\pm. \quad (2.39)$$

It is not difficult to see (e.g. [46, Section 2.1]) that

$$\text{dom}(A) := \{y \in \text{dom}((A_{\min}^+)^*) \oplus ((A_{\min}^-)^*) : y(+0) = y(-0), y'(+0) = y'(-0)\}. \quad (2.40)$$

Put $\hat{A}_{\min}^\pm := \mathcal{F}_\pm A_{\min}^\pm \mathcal{F}_\pm^{-1}$ and recall that $\hat{A}_0^\pm := \mathcal{F}_\pm A_0^\pm \mathcal{F}_\pm^{-1}$ is the operator of multiplication by t in the space $L^2(\mathbb{R}, d\Sigma_{N\pm}(t))$, i.e., $\hat{A}_0^\pm = Q_{\Sigma_{N\pm}}$.

Let functions $f \in L^2(\mathbb{R})$ and $f_\pm \in L^2(\mathbb{R}_\pm)$ be such that $f = f_+ + f_-$. Denote $g^\pm(t) := (\mathcal{F}_\pm f_\pm)(t)$. From (2.39) we get

$$(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)(\pm 0) = \int_{\mathbb{R}} \frac{g^\pm(t) d\Sigma_{N\pm}(t)}{t - \lambda}. \quad (2.41)$$

Since $\mathcal{R}_{\hat{A}_0^\pm}(\lambda)g^\pm(t) = g^\pm(t)(t - \lambda)^{-1}$, we see that

$$y_\pm(0) = \int_{\mathbb{R}} (\mathcal{F}_\pm y_\pm)(t) d\Sigma_{N\pm}(t) \quad \text{for all } y_\pm \in \text{dom}(A_0^\pm), \quad (2.42)$$

$$\text{and } \hat{A}_{\min}^\pm = Q_{\Sigma_\pm} \upharpoonright \text{dom}(\hat{A}_{\min}^\pm),$$

$$\text{dom}(\hat{A}_{\min}^\pm) = \{\hat{y}_\pm \in \text{dom}(Q_{\Sigma_{N\pm}}) : \int_{\mathbb{R}} \hat{y}_\pm(t) d\Sigma_{N\pm}(t) = 0\}.$$

That is, $\hat{A}_{\min}^\pm = T_{\Sigma_{N\pm}}$.

It follows from (2.38) that $(\mathcal{R}_{A_0^\pm}(\lambda)f_\pm)(\pm 0) = \pm \int_0^{\pm\infty} \psi_\pm(x, \lambda)f_\pm(x)dx$ for $\lambda \notin \mathbb{R}$. From this and (2.41), we get

$$(\mathcal{F}_\pm \psi_\pm(\cdot, \lambda))(t) = \frac{1}{t - \lambda} \in L^2(\mathbb{R}, d\Sigma_{N\pm}). \quad (2.43)$$

Let $y_\pm(x) \in \text{dom}((A_{\min}^\pm)^*)$. Then, by the von Neumann formula,

$$y_\pm(t) = y_{0\pm}(t) + c_1 \psi_\pm(t, i) + c_2 \psi_\pm(t, -i), \quad (2.44)$$

where $y_{0\pm}(t) \in \text{dom}(A_{\min}^\pm)$ and $c_1, c_2 \in \mathbb{C}$ are certain constants. Therefore (2.32) yields

$$y_\pm(0) = c_1 M_{N\pm}(i) + c_2 M_{N\pm}(-i) = c_1 \int_{\mathbb{R}} \frac{1}{t - i} d\Sigma_{N\pm}(t) + c_2 \int_{\mathbb{R}} \frac{1}{t + i} d\Sigma_{N\pm}(t).$$

This, (2.43), and (2.42) implies that (2.42) holds for all $y_\pm(x) \in \text{dom}((A_{\min}^\pm)^*)$. Taking (2.36) and (2.3) into account, we get

$$y_\pm(0) = \int_{\mathbb{R}} (\mathcal{F}_\pm y_\pm)(t) d\Sigma_\pm(t) = \Gamma_1^{\Sigma_{N\pm}, C_{N\pm}} \mathcal{F}_\pm y_\pm. \quad (2.45)$$

Further, by (2.44), $y'_\pm(0) = -c_1 - c_2$. On the other hand, it follows from $\widehat{A}_{\min}^\pm = T_{\Sigma_{N\pm}}$ and (2.3) that $\Gamma_0^{\Sigma_{N\pm}} \mathcal{F}y_{0\pm} = 0$ and $\Gamma_0^{\Sigma_{N\pm}}(t - \lambda)^{-1} = 1$. Hence,

$$\Gamma_0^{\Sigma_{N\pm}} \mathcal{F}y_\pm = c_1 \Gamma_0^{\Sigma_{N\pm}} \frac{1}{t - i} + c_2 \Gamma_0^{\Sigma_{N\pm}} \frac{1}{t + i} = c_1 + c_2 = -y'_\pm(0). \quad (2.46)$$

Combining (2.40), (2.45), and (2.46), we get (2.37). \square

Remark 2.7. Since the operators $T_{\Sigma_{N\pm}}$ are simple (see [57, Proposition 7.9]), in passing it is proved that so are the operators A_{\min}^\pm and A_{\min} . This proof of simplicity works in general case of Sturm-Liouville operator with one turning point described in the beginning of this section. Formally, it removes extra smoothness assumptions on the coefficient p imposed in [30]. But actually it is just another version of the proof of [30, Theorem 3] since the essence of both the proofs is based on Kreins criterion for simplicity [53, Section 1.3].

3 Point and essential spectra of the model operator \widehat{A} and of indefinite Sturm-Liouville operators

3.1 Point spectrum of the model operator

The main result of this section and of the paper is a description of the point spectrum and algebraic multiplicities of eigenvalues of $\widehat{A}\{\Sigma_+, C_+, \Sigma_-, C_-\}$.

First, to classify eigenvalues of the operator T_Σ^* defined in Subsection 2.1, we introduce the following mutually disjoint sets:

$$\begin{aligned} \mathfrak{A}_0(\Sigma) &= \left\{ \lambda \in \sigma_c(Q_\Sigma) : \int_{\mathbb{R}} |t - \lambda|^{-2} d\Sigma(t) = \infty \right\}, \\ \mathfrak{A}_r(\Sigma) &= \left\{ \lambda \notin \sigma_p(Q_\Sigma) : \int_{\mathbb{R}} |t - \lambda|^{-2} d\Sigma(t) < \infty \right\}, \\ \mathfrak{A}_p(\Sigma) &= \sigma_p(Q_\Sigma). \end{aligned} \quad (3.1)$$

Observe that $\mathbb{C} = \mathfrak{A}_0(\Sigma) \cup \mathfrak{A}_r(\Sigma) \cup \mathfrak{A}_p(\Sigma)$ and

$$\mathfrak{A}_0(\Sigma) = \{ \lambda \in \mathbb{C} : \ker(T_\Sigma^* - \lambda I) = \{0\} \}, \quad (3.2)$$

$$\mathfrak{A}_r(\Sigma) = \{ \lambda \in \mathbb{C} : \ker(T_\Sigma^* - \lambda I) = \{c(t - \lambda)^{-1}, c \in \mathbb{C}\} \}, \quad (3.3)$$

$$\mathfrak{A}_p(\Sigma) = \{ \lambda \in \mathbb{C} : \ker(T_\Sigma^* - \lambda I) = \{c\chi_{\{\lambda\}}(t), c \in \mathbb{C}\} \}.$$

In this section we denote for brevity $\Gamma_0^\pm := \Gamma_0^{\Sigma_\pm}$, $\Gamma_1^\pm := \Gamma_1^{\Sigma_\pm, C_\pm}$, where $\Gamma_0^{\Sigma_\pm}$, $\Gamma_1^{\Sigma_\pm, C_\pm}$ are linear mappings from $\text{dom}(T_{\Sigma_\pm}^*)$ to \mathbb{C} defined by (2.3).

In this paper, for fixed $\lambda \in \mathbb{R}$, the notation $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j}$ means the function that is equal to 0 at $t = \lambda$ and $\frac{1}{(t - \lambda)^j}$ for $t \neq \lambda$. If $\lambda \notin \mathbb{R}$, then $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j}$ means just $\frac{1}{(t - \lambda)^j}$. In what follows the functions $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j}$ and jump discontinuities of Σ play an essential role. Note that the set of jump discontinuities of Σ coincides with $\mathfrak{A}_p(\Sigma) = \sigma_p(Q_\Sigma)$. If $\lambda \in \mathbb{R} \setminus \mathfrak{A}_p(\Sigma)$, then $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j}$ and $\frac{1}{(t - \lambda)^j}$ belong

to the same class of $L^2(\mathbb{R}, d\Sigma)$ and any of these two notations can be used. We also use notation $d\Sigma(\{\lambda\}) := \Sigma(\lambda + 0) - \Sigma(\lambda - 0)$.

For the sake of simplicity, we start from the case when

$$\int_{\mathbb{R}} (1 + |t|)^{-1} d\Sigma_{\pm} < \infty \quad \text{and} \quad C_{\pm} = \int_{\mathbb{R}} t(1 + t^2)^{-1} d\Sigma_{\pm}, \quad (3.4)$$

(which arises, in particular, in Section 2.3) and then consider the general case.

Theorem 3.1. *Let Σ_{\pm} be nondecreasing scalar functions satisfying (2.1) and let C_{\pm} be real constants. Assume also that conditions (3.4) are fulfilled. Then the following statements describe the point spectrum of the operator $\widehat{A} = \widehat{A}\{\Sigma_+, C_+, \Sigma_-, C_-\}$.*

1) *If $\lambda \in \mathfrak{A}_0(\Sigma_+) \cup \mathfrak{A}_0(\Sigma_-)$, then $\lambda \notin \sigma_p(\widehat{A})$.*

2) *If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$, then*

- (i) *λ is an eigenvalue of \widehat{A} ; the geometric multiplicity of λ equals 1;*
- (ii) *the eigenvalue λ is simple (i.e., the algebraic and geometric multiplicities are equal to 1) if and only if at least one of the following conditions is not fulfilled:*

$$d\Sigma_-(\{\lambda\}) = d\Sigma_+(\{\lambda\}), \quad (3.5)$$

$$\int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{|t - \lambda|^2} d\Sigma_+(t) < \infty, \quad (3.6)$$

$$\int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{|t - \lambda|^2} d\Sigma_-(t) < \infty; \quad (3.7)$$

- (iii) *if conditions (3.5), (3.6) and (3.7) hold true, then the algebraic multiplicity of λ equals the greatest number k ($k \in \{2, 3, 4, \dots\} \cup \{+\infty\}$) such that the conditions*

$$\int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{|t - \lambda|^{2j}} d\Sigma_-(t) < \infty, \quad \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{|t - \lambda|^{2j}} d\Sigma_+(t) < \infty, \quad (3.8)$$

$$\int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{(t - \lambda)^{j-1}} d\Sigma_-(t) = \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{(t - \lambda)^{j-1}} d\Sigma_+(t), \quad (3.9)$$

are fulfilled for all natural j such that $2 \leq j \leq k - 1$ (in particular, $k = 2$ if at least one of conditions (3.8), (3.9) is not fulfilled for $j = 2$).

3) *Assume that $\lambda \in \mathfrak{A}_r(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$. Then $\lambda \in \sigma_p(\widehat{A})$ if and only if*

$$\int_{\mathbb{R}} \frac{1}{t - \lambda} d\Sigma_+(t) = \int_{\mathbb{R}} \frac{1}{t - \lambda} d\Sigma_-(t). \quad (3.10)$$

If (3.10) holds true, then the geometric multiplicity of λ is 1, and the algebraic multiplicity is the greatest number k ($k \in \{1, 2, 3, \dots\} \cup \{+\infty\}$) such that the conditions

$$\int_{\mathbb{R}} \frac{1}{|t - \lambda|^{2j}} d\Sigma_-(t) < \infty, \quad \int_{\mathbb{R}} \frac{1}{|t - \lambda|^{2j}} d\Sigma_+(t) < \infty, \quad (3.11)$$

$$\int_{\mathbb{R}} \frac{1}{(t - \lambda)^j} d\Sigma_-(t) = \int_{\mathbb{R}} \frac{1}{(t - \lambda)^j} d\Sigma_+(t) \quad (3.12)$$

are fulfilled for all $j \in \mathbb{N}$ such that $1 \leq j \leq k$.

4. If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$ or $\lambda \in \mathfrak{A}_p(\Sigma_-) \cap \mathfrak{A}_r(\Sigma_+)$, then $\lambda \notin \sigma_p(\widehat{A})$.

Proof. A vector $y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix} \in L^2(\mathbb{R}, d\Sigma_+) \oplus L^2(\mathbb{R}, d\Sigma_-)$ is a solution of the equation $\widehat{A}y = \lambda y$ if and only if

$$y \in \ker(T_{\Sigma_+}^* - \lambda I) \oplus \ker(T_{\Sigma_-}^* - \lambda I) \quad \text{and} \quad y \in \text{dom}(\widehat{A}).$$

Recall that $h = \begin{pmatrix} h_- \\ h_+ \end{pmatrix} \in \text{dom}(T_{\Sigma_+}^*) \oplus \text{dom}(T_{\Sigma_-}^*)$ belongs to $\text{dom}(\widehat{A})$ if and only if

$$\Gamma_0^- h_- = \Gamma_0^+ h_+, \quad \Gamma_1^- h_- = \Gamma_1^+ h_+. \quad (3.13)$$

It follows from (2.3) that

$$\Gamma_1^\pm h_\pm = C_\pm \Gamma_0^\pm h_\pm + \int_{\mathbb{R}} \left(h_\pm(t) - \frac{t \Gamma_0^\pm h_\pm}{t^2 + 1} \right) d\Sigma_\pm(t), \quad h_\pm \in \text{dom}(T_{\Sigma_\pm}^*). \quad (3.14)$$

(3.4) and (2.3) yield $h_\pm(t) \in L^1(\mathbb{R}, d\Sigma_\pm)$ for arbitrary $h_\pm(t) \in \text{dom}(T_{\Sigma_\pm}^*)$, and using (3.14), we obtain

$$\Gamma_1^\pm y_\pm = \int_{\mathbb{R}} y_\pm(t) d\Sigma_\pm(t). \quad (3.15)$$

If $\lambda \in \mathfrak{A}_r$, then $\frac{1}{t-\lambda} \in L^2(\mathbb{R}, d\Sigma_\pm)$ and (2.3) (or even simpler (2.5)) yields that $\frac{1}{t-\lambda} \in \text{dom}(T_{\Sigma_\pm}^*)$ and

$$\Gamma_0^\pm \frac{1}{t-\lambda} = 1. \quad (3.16)$$

The function $\chi_{\{\lambda\}}(t)$, $\lambda \in \mathbb{R}$, is a nonzero vector in $L^2(\mathbb{R}, d\Sigma_\pm)$ exactly when $\lambda \in \mathfrak{A}_p$; in this case,

$$\Gamma_0^\pm \chi_{\{\lambda\}} = 0, \quad \Gamma_1^\pm \chi_{\{\lambda\}} = \int_{\{\lambda\}} d\Sigma_\pm(t) = d\Sigma_\pm(\{\lambda\}). \quad (3.17)$$

1) Suppose $\widehat{A}y = \lambda y$ and consider the case $\ker(T_{\Sigma_-}^* - \lambda) = \{0\}$ (the case $\ker(T_{\Sigma_+}^* - \lambda) = \{0\}$ is analogous). Then $y_- = 0$ and, by (3.13), we get $\Gamma_0^+ y_+ = 0$, $\Gamma_1^+ y_+ = 0$. Hence $y_+ \in \text{dom}(Q_{\Sigma_+})$ (see (2.3)), and $Q_{\Sigma_+} y_+ = \lambda y_+$. This implies $y_+(t) = c_1 \chi_{\{\lambda\}}(t)$, $c_1 \in \mathbb{C}$. On the other hand, $0 = \Gamma_1^+ y_+(t) = \int_{\mathbb{R}} y_+(t) d\Sigma_+(t)$. Thus $c_1 = 0$ and $y_+ = 0$ a.e. with respect to the measure $d\Sigma_+$.

2) Let $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$. By (3.3), we have

$$y(t) = \begin{pmatrix} c_1^- \chi_{\{\lambda\}}(t) \\ c_1^+ \chi_{\{\lambda\}}(t) \end{pmatrix}, \quad c_1^\pm \in \mathbb{C}.$$

Since $\lambda \in \mathfrak{A}_p(\Sigma_\pm)$, we see that $\lambda \in \mathbb{R}$ and $d\Sigma_\pm(\{\lambda\}) \neq 0$. Taking into account (3.17), we see that system (3.13) is equivalent to $c_1^- d\Sigma_-(\{\lambda\}) = c_1^+ d\Sigma_+(\{\lambda\})$. Therefore the geometric multiplicity of λ equals 1 and

$$\widehat{y}_0 = \begin{pmatrix} \frac{1}{d\Sigma_-(\{\lambda\})} \chi_{\{\lambda\}}(t) \\ \frac{1}{d\Sigma_+(\{\lambda\})} \chi_{\{\lambda\}}(t) \end{pmatrix} \quad \text{is one of corresponding eigenvectors of } \widehat{A}. \quad (3.18)$$

Let $y_1 = \begin{pmatrix} y_1^- \\ y_1^+ \end{pmatrix}$ and $\widehat{A}y_1 - \lambda y_1 = y_0$. By (2.5), we have

$$\begin{pmatrix} ty_1^-(t) \\ ty_1^+(t) \end{pmatrix} - \begin{pmatrix} \Gamma_0^- y_1^- \\ \Gamma_0^+ y_1^+ \end{pmatrix} - \lambda \begin{pmatrix} y_1^-(t) \\ y_1^+(t) \end{pmatrix} = y_0 .$$

Thus,

$$(t - \lambda)y_1^\pm(t) = \frac{1}{d\Sigma_\pm(\{\lambda\})} \chi_{\{\lambda\}}(t) + \Gamma_0^\pm y_1^\pm .$$

Choosing $t = \lambda$, we obtain

$$\Gamma_0^\pm y_1^\pm = -\frac{1}{d\Sigma_\pm(\{\lambda\})} \neq 0. \quad (3.19)$$

Therefore,

$$y_1^\pm = -\frac{1}{d\Sigma_\pm(\{\lambda\})} \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} + c_2^\pm \chi_{\{\lambda\}}(t) , \quad (3.20)$$

where $c_2^\pm \in \mathbb{C}$. The conditions $y_1^+ \in L^2(\mathbb{R}, d\Sigma_+)$ and $y_1^- \in L^2(\mathbb{R}, d\Sigma_-)$ are equivalent to (3.6) and (3.7), respectively.

Assume that (3.6) and (3.7) are fulfilled. By (3.15), we have

$$\Gamma_1^\pm y_1^\pm = -\frac{1}{d\Sigma_\pm(\{\lambda\})} \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t - \lambda} d\Sigma_\pm(t) + c_2^\pm d\Sigma_\pm(\{\lambda\}).$$

The latter and (3.19) implies that $\widehat{y}_1 \in \text{dom}(\widehat{A})$ if and only if the conditions (3.5) and

$$\begin{aligned} & -\frac{1}{d\Sigma_-(\{\lambda\})} \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t - \lambda} d\Sigma_-(t) + c_2^- d\Sigma_-(\{\lambda\}) = \\ & = -\frac{1}{d\Sigma_+(\{\lambda\})} \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t - \lambda} d\Sigma_+(t) + c_2^+ d\Sigma_+(\{\lambda\}) \end{aligned} \quad (3.21)$$

are fulfilled. Thus, the quotient space $\ker(\widehat{A} - \lambda)^2 / \ker(\widehat{A} - \lambda) \neq \{0\}$ if and only if the conditions (3.5), (3.6), and (3.7) are satisfied. In this case, generalized eigenvectors of first order y_1 have the form (3.20) with constants c_2^\pm such that (3.21) holds.

Assume that all condition mentioned above are satisfied. Then $\dim \ker(\widehat{A} - \lambda)^2 / \ker(\widehat{A} - \lambda) = 1$ and one of generalized eigenvectors of first order is given by the constants

$$c_2^\pm = -\alpha_1^2 \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t - \lambda} d\Sigma_\mp(t),$$

where $\alpha_1 := \frac{1}{d\Sigma_-(\{\lambda\})} = \frac{1}{d\Sigma_+(\{\lambda\})}$.

If $y_2 = \begin{pmatrix} y_2^- \\ y_2^+ \end{pmatrix}$ and $\widehat{A}y_2 - \lambda y_2 = y_1$, then

$$(t - \lambda)y_2^\pm(t) = y_1^\pm(t) + \Gamma_0^\pm y_2^\pm = -\alpha_1 \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} + c_2^\pm \chi_{\{\lambda\}}(t) + \Gamma_0^\pm y_2^\pm . \quad (3.22)$$

For $t = \lambda$ we have

$$\Gamma_0^\pm y_2^\pm = -c_2^\pm. \quad (3.23)$$

Consequently,

$$y_2^\pm = -\alpha_1 \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^2} - c_2^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t-\lambda} + c_3^\pm \chi_{\{\lambda\}}(t), \quad c_3^\pm \in \mathbb{C}. \quad (3.24)$$

By (3.23), conditions (3.13) for y_2 has the form

$$c_2^- = c_2^+, \quad (3.25)$$

$$\begin{aligned} & -\alpha_1 \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{(t-\lambda)^2} d\Sigma_-(t) - c_2^- \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t-\lambda} d\Sigma_-(t) + c_3^- \alpha_1^{-1} = \\ & = -\alpha_1 \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{(t-\lambda)^2} d\Sigma_+(t) - c_2^+ \int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{t-\lambda} d\Sigma_+(t) + c_3^+ \alpha_1^{-1}. \end{aligned}$$

Thus y_2 exists if and only if $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^2} \in L^2(\mathbb{R}, d\Sigma_\pm)$ and (3.25) is fulfilled. This is equivalent (3.8) and (3.9) for $j = 2$.

Continuing this line of reasoning, we obtain part 2) of the theorem.

3) The idea of the proof for part 3) is similar to that of part 2), but calculations are simpler.

Let $\lambda \in \mathfrak{A}_r(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$. Then $y(t) = \begin{pmatrix} c_1^- \frac{1}{t-\lambda} \\ c_1^+ \frac{1}{t-\lambda} \end{pmatrix}$. Hence (3.13) has the form

$$c_1^- = c_1^+, \quad c_1^- \int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_-(t) = c_1^+ \int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_+(t).$$

Consequently λ is an eigenvalue of \hat{A} if and only if (3.10) holds true; in this case the geometric multiplicity is 1 and $y_0 = \begin{pmatrix} \frac{1}{t-\lambda} \\ \frac{1}{t-\lambda} \end{pmatrix}$ is a corresponding eigenvector of \hat{A} .

Let $\hat{A}y_1 - \lambda y_1 = y_0$ where $y_1 = \begin{pmatrix} y_1^- \\ y_1^+ \end{pmatrix}$. Then

$$(t-\lambda)y_1^\pm(t) = \frac{1}{t-\lambda} + c_2^\pm, \quad c_2^\pm = \Gamma_0^\pm y_1^\pm.$$

Therefore $y_1^\pm = \frac{1}{(t-\lambda)^2} + \frac{c_2^\pm}{t-\lambda}$. The case $y_1^\pm \in L^2(\mathbb{R}, d\Sigma_\pm)$ is characterized by (3.11) with $j = 2$. Conditions (3.13) become

$$\begin{aligned} \int_{\mathbb{R}} \left(\frac{1}{(t-\lambda)^2} + \frac{c_2^-}{t-\lambda} \right) d\Sigma_-(t) &= \int_{\mathbb{R}} \left(\frac{1}{(t-\lambda)^2} + \frac{c_2^+}{t-\lambda} \right) d\Sigma_+(t), \\ c_2^- &= c_2^+. \end{aligned}$$

Taking into account (3.10), we see that the generalized eigenvector \widehat{y}_1 exists if and only if conditions (3.11), (3.12) are satisfied for $j = 2$. Continuing this line of reasoning, we obtain part 3) of the theorem.

4) Suppose $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$ (the case $\lambda \in \mathfrak{A}_p(\Sigma_-) \cap \mathfrak{A}_r(\Sigma_+)$ is similar). Then $\widehat{y}(t) = \begin{pmatrix} c_1^- \frac{1}{t-\lambda} \\ c_1^+ \chi_{\{\lambda\}}(t) \end{pmatrix}$ and (3.13) has the form

$$c_1^- = 0, \quad c_1^- \int_{\mathbb{R}} \frac{1}{t-\lambda} d\Sigma_-(t) = c_1^+ d\Sigma_+(\{\lambda\}).$$

Thus $c_1^- = c_1^+ = 0$ and $\lambda \notin \sigma_p(\widehat{A})$. □

Now we consider the general case when the functions Σ_{\pm} satisfy (2.1) and C_{\pm} are arbitrary real constants.

Lemma 3.2. *Let $k \in \mathbb{N}$ and let one of the following two assumptions be fulfilled:*

(a) $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or

(b) $\lambda \in \mathbb{R}$, $d\Sigma_+(\{\lambda\}) = d\Sigma_-(\{\lambda\})$, $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k} \in L^2(\mathbb{R}, d\Sigma_+)$,
and $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k} \in L^2(\mathbb{R}, d\Sigma_-)$.

Then $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k} \in \text{dom}(T_{\Sigma_+}^*)$, $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k} \in \text{dom}(T_{\Sigma_-}^*)$, and the following two statements are equivalent:

(i) $\Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k} = \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^k}$;

(ii) $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \Phi^{(k-1)}(\lambda + i\varepsilon) = 0$, where the function Φ is defined by

$$\Phi := M_{\Sigma_+, C_+} - M_{\Sigma_-, C_-} \text{ and } \Phi^{(j)} \text{ is its } j\text{-th derivative } (\Phi^{(0)} = \Phi).$$

If, additionally, $\lambda \notin \sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-})$, then statements (i) and (ii) are equivalent to

(iii) the function Φ is analytic in a certain neighborhood of λ and

$\Phi^{(k-1)}(\lambda) = 0$. (If $M_{\Sigma_+, C_+} - M_{\Sigma_-, C_-}$ is defined in a punctured neighborhood of λ and has a removable singularity at λ , then we assume that Φ is analytically extended over λ .)

Proof. We assume here and below that $j \in \mathbb{N}$.

First note that if $\lambda \notin \sigma_{\text{ess}}(Q_{\Sigma_{\pm}})$, then $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^j} \in L^2(\mathbb{R}, d\Sigma_{\pm})$ for any $j \in \mathbb{N}$, and using the definition of $\text{dom}(T_{\Sigma_{\pm}}^*)$, we see that $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^j} \in \text{dom}(T_{\Sigma_{\pm}}^*)$ for any j .

Generally, the last statement is not true for $\lambda \in \sigma_{\text{ess}}(Q_{\Sigma_{\pm}})$. But under assumptions of the lemma, we have

$$\int_{\mathbb{R} \setminus \{\lambda\}} \frac{1}{|t-\lambda|^{2j}} d\Sigma_{\pm}(t) < \infty \quad (3.26)$$

for $j = k$. Taking into account the first assumption in (2.1), we see that (3.26) is valid for all $j \leq k$. The latter implies that $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^j} \in \text{dom}(T_{\Sigma_{\pm}}^*)$ for all $j \leq k$. Moreover, $\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^j} \in \text{dom}(Q_{\Sigma_{\pm}})$ if $2 \leq j \leq k$ (assuming $k \geq 2$). Therefore,

$$\Gamma_0^{\pm} \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t-\lambda)^j} = 0, \quad 2 \leq j \leq k.$$

The last statement does not hold in the case $j = 1$. Using (2.5), one has

$$\Gamma_0^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} = 1. \quad (3.27)$$

Eqs. (2.3) (see also (3.14)) allow us to conclude that

$$\Gamma_1^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} = C_\pm + \int_{\mathbb{R}} \left(\frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} - \frac{t}{t^2 + 1} \right) d\Sigma_\pm(t), \quad (3.28)$$

$$\Gamma_1^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j} = \int_{\mathbb{R}} \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j} d\Sigma_\pm(t) \quad \text{if } 2 \leq j \leq k. \quad (3.29)$$

If $\lambda \notin \sigma(Q_{\Sigma_\pm})$ (in particular, if $\lambda \notin \mathbb{R}$), then (2.4) shows that

$$\Gamma_1^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j} = \Gamma_1^\pm \frac{1}{(t - \lambda)^j} = (j - 1)! M_{\Sigma_\pm, C_\pm}^{(j-1)}(\lambda).$$

This prove the equivalence of (i), (ii), and (iii) for the case when $\lambda \notin \sigma(Q_{\Sigma_+}) \cup \sigma(Q_{\Sigma_-})$ (this simplest case explains the crux of the lemma).

Consider the case $\lambda \in \sigma(Q_{\Sigma_+}) \cup \sigma(Q_{\Sigma_-})$ and $\lambda \notin \sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-})$. The assumptions of the lemma state that $d\Sigma_+(\{\lambda\}) = d\Sigma_-(\{\lambda\})$. So λ is an isolated eigenvalue of both the operators Q_{Σ_+} and Q_{Σ_-} and is an isolated jump discontinuity of Σ_+ and Σ_- . This and $d\Sigma_+(\{\lambda\}) = d\Sigma_-(\{\lambda\})$ imply that Φ has a removable singularity at λ and can be considered as an analytic function in a certain neighborhood of λ . Moreover,

$$(k - 1)! \Phi^{(k-1)}(\lambda) = \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^k} - \Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^k},$$

and (i) \Leftrightarrow (ii) \Leftrightarrow (iii) is shown again.

Now let assumption (b) be satisfied and let $\lambda \in \sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-})$. Then the function Φ is not analytic in λ , but the limit in statement (ii) exists and

$$(k - 1)! \lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \Phi^{(k-1)}(\lambda + i\varepsilon) = \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^k} - \Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^k}. \quad (3.30)$$

Indeed, taking $d\Sigma_+(\{\lambda\}) = d\Sigma_-(\{\lambda\})$ into account, we get $\Phi(z) = C_+ - C_- + \mathcal{I}_\infty(z) + \mathcal{I}_\lambda(z)$, where

$$\begin{aligned} \mathcal{I}_\infty(z) &:= \int_{\mathbb{R} \setminus [\lambda - \delta, \lambda + \delta]} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) (d\Sigma_+(t) - d\Sigma_-(t)) , \\ \mathcal{I}_\lambda(z) &:= \int_{[\lambda - \delta, \lambda] \cup (\lambda, \lambda + \delta]} \left(\frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right) (d\Sigma_+(t) - d\Sigma_-(t)) , \end{aligned}$$

and δ is any fixed positive number. The function $\mathcal{I}_\infty(z)$ is analytic at λ . Formula (3.26) is valid for $j \leq k$ and allows us to apply Lebesgue's dominated convergence theorem to the limit $\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \mathcal{I}_\lambda^{(j-1)}(\lambda + i\varepsilon)$.

As a result, we see that (3.28) implies (3.30) for $k = 1$ and (3.29) implies (3.30) for $k \geq 2$. \square

Theorem 3.3. *Let $\hat{A} = \hat{A}\{\Sigma_+, C_+, \Sigma_-, C_-\}$, where the functions $d\Sigma_\pm$ satisfy (2.1) and C_\pm are certain real constants. Then the following statements hold:*

1) If $\lambda \in \mathfrak{A}_0(\Sigma_+) \cup \mathfrak{A}_0(\Sigma_-)$, then $\lambda \notin \sigma_p(\hat{A})$.

2) If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$, then

- (i) λ is an eigenvalue of \hat{A} ; the geometric multiplicity of λ equals 1;
- (ii) the eigenvalue λ is simple if and only if at least one of conditions (3.5), (3.6), (3.7) is not fulfilled;
- (iii) if conditions (3.5), (3.6) and (3.7) hold true, then the algebraic multiplicity of λ equals the greatest number k ($k \in \{2, 3, 4, \dots\} \cup \{+\infty\}$) such that conditions (3.8) and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \Phi^{(j-2)}(\lambda + i\varepsilon) = 0 \quad (\text{the function } \Phi \text{ is defined in Lemma 3.2}), \quad (3.31)$$

are fulfilled for all $j \in \mathbb{N}$ such that $2 \leq j \leq k-1$ (in particular, $k=2$ if at least one of conditions (3.8), (3.31) is not fulfilled for $j=2$).

3) Assume that $\lambda \in \mathfrak{A}_r(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$. Then $\lambda \in \sigma_p(\hat{A})$ if and only if

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \Phi(\lambda + i\varepsilon) = 0. \quad (3.32)$$

If (3.32) holds true, then the geometric multiplicity of λ is 1, and the algebraic multiplicity is the greatest number k ($1 \leq k \leq \infty$) such that the conditions (3.11) and

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon \in \mathbb{R}}} \Phi^{(j-1)}(\lambda + i\varepsilon) = 0 \quad (3.33)$$

are fulfilled for all $j \in \mathbb{N}$ such that $1 \leq j \leq k$.

4. If $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_r(\Sigma_-)$ or $\lambda \in \mathfrak{A}_p(\Sigma_-) \cap \mathfrak{A}_r(\Sigma_+)$, then $\lambda \notin \sigma_p(\hat{A})$.

Proof. The proof is similar to that of Theorem 3.1, but some technical complications appear. Namely, (3.15) is not valid whenever any of conditions in (3.4) is not satisfied.

We have to use (3.14), which is valid in the general case. Note that (3.17) holds true. In the case $\lambda \in \mathfrak{A}_r(\Sigma_\pm)$, (3.16) holds also. When $\lambda \in \sigma_p(Q_{\Sigma_\pm})$, Eq. (3.16) should be changed to (3.27).

The proof of statements 1) and 4) remains the same.

2) Let $\lambda \in \mathfrak{A}_p(\Sigma_+) \cap \mathfrak{A}_p(\Sigma_-)$. As before, we see that λ is an eigenvalue of \hat{A} with geometric multiplicity 1 and one of corresponding eigenvectors has the form (3.18).

Let $y_1 = \begin{pmatrix} y_1^- \\ y_1^+ \end{pmatrix}$ and $\hat{A}y_1 - \lambda y_1 = y_0$. In the same way, we get (3.19), (3.20) as well as the fact that the conditions $y_1^+ \in L^2(\mathbb{R}, d\Sigma_+)$ and $y_1^- \in L^2(\mathbb{R}, d\Sigma_-)$ are equivalent to (3.6) and (3.7), respectively. If (3.6) and (3.7) are fulfilled, we obtain

$$\Gamma_1^\pm y_1^\pm = -\frac{1}{d\Sigma_\pm(\{\lambda\})} \Gamma_1^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} + c_2^\pm d\Sigma_\pm(\{\lambda\}).$$

The latter and (3.19) implies that $y_1 \in \text{dom}(\hat{A})$ if and only if conditions (3.5) and

$$-\frac{1}{d\Sigma_-(\{\lambda\})} \Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} + c_2^- d\Sigma_-(\{\lambda\}) = -\frac{1}{d\Sigma_+(\{\lambda\})} \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda} + c_2^+ d\Sigma_+(\{\lambda\}) \quad (3.34)$$

are fulfilled. Thus, generalized eigenvectors of first order exist if and only if conditions (3.5), (3.6), and (3.7) are satisfied. In this case, y_1 has the form (3.20) with constants c_2^\pm such that (3.34) holds. In particular, the constants

$$c_2^\pm = -\alpha_1^2 \Gamma_1^\mp \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{t - \lambda}, \quad (3.35)$$

give a generalized eigenvector (as before, $\alpha_1 = \frac{1}{d\Sigma_-(\{\lambda\})} = \frac{1}{d\Sigma_+(\{\lambda\})}$).

Let $y_2 = \begin{pmatrix} y_2^- \\ y_2^+ \end{pmatrix}$ and $\widehat{A}y_2 - \lambda y_2 = y_1$. Then (3.22), (3.23), and (3.24) have to be fulfilled with c_2^\pm given by (3.35). So y_2^\pm belong to $L^2(\mathbb{R}, d\Sigma_\pm)$ if and only if (3.8) is satisfied for $j = 2$. Conditions (3.13) are equivalent to $c_2^- = c_2^+$ and

$$\begin{aligned} & -\alpha_1 \Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^2} - c_2^- \Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{t - \lambda} + c_3^- \alpha_1^{-1} = \\ & = -\alpha_1 \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^2} - c_2^+ \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{t - \lambda} + c_3^+ \alpha_1^{-1}. \end{aligned}$$

Thus y_2 exists if and only if, for $j = 2$, conditions (3.8) and

$$\Gamma_1^- \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^{j-1}} = \Gamma_1^+ \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}}{(t - \lambda)^{j-1}} \quad (3.36)$$

are fulfilled. By Lemma 3.2, (3.36) is equivalent to (3.31) with $j = 2$. Continuing this line of reasoning, we obtain parts **2)** and **3)** of the theorem. \square

Remark 3.4. **(1)** In the last theorem, the conditions that determine the algebraic multiplicities are given in the terms of the function $\Phi = M_{\Sigma_+, C_+} - M_{\Sigma_-, C_-}$, so in the terms of abstract Weyl functions M_{Σ_\pm, C_\pm} . Using Lemma 3.2 and (3.28), (3.29), Theorem 3.3 can be easily rewritten in terms of the spectral measures Σ_\pm , but this makes the answer longer due to the different forms of $\Gamma_1^\pm \frac{\chi_{\mathbb{R} \setminus \{\lambda\}}(t)}{(t - \lambda)^j}$ for the cases $j = 1$ and $j \geq 2$, see (3.28) and (3.29). In the case when assumptions (3.4) are fulfilled, (3.28) can be written in the form of (3.29) and we get Theorem 3.1.

(2) Note that eigenvalues of \widehat{A} that belong to $\rho(Q_{\Sigma_+} \oplus Q_{\Sigma_+})$ can be found in the terms of M_{Σ_\pm, C_\pm} using [21] (and, perhaps, [18]), see the next section. Algebraic multiplicities of eigenvalues in $\rho(Q_{\Sigma_+} \oplus Q_{\Sigma_+})$ can be found using Krein's resolvent formula (see [21, 22] for a convenient abstract form), root subspaces for eigenvalues in $\rho(Q_{\Sigma_+} \oplus Q_{\Sigma_+})$ were found in [19]. Theorem 3.3 has some common points with [12], where the abstract Weyl function was used to find eigenvalues of a self-adjoint operator. But the approach of the present paper goes in the backward direction: we use the spectral measures $d\Sigma_\pm$ and the functional model to find eigenvalues and root subspaces and then, using Lemma 3.2, return to the answer in the terms of the abstract Weyl functions given in Theorem 3.3.

(3) Various generalizations of (R)-functions and their functional models were considered in [25, 34]. These results were applied to certain classes of regular Sturm-Liouville problems in [26, 10, 27].

3.2 Essential and discrete spectra of the model operator and of indefinite Sturm-Liouville operators

Besides the symmetry condition $\sigma(T) = \sigma(T)^*$ the spectrum of a J-self-adjoint operator can be fairly arbitrary (see [54]). An example of a differential operator with a "wild" spectrum was given in [4, 2].

Example 3.5. Consider the operator A in $L^2[-1, 1]$ associated with the differential expression

$$(\operatorname{sgn} x) ((\operatorname{sgn} x)y')'$$

and boundary conditions $y(-1) = 0 = y(1)$. More precisely, $Ay = -y''$,

$$\operatorname{dom}(A) = \{y \in W_2^2(-1, 0) \oplus W_2^2(0, 1) : y(-0) = y(+0), y'(-0) = -y'(+0) \text{ and } y(-1) = 0 = y(1)\}.$$

The operator A is J -self-adjoint with J given by $Jf(x) = (\operatorname{sgn} x)f(x)$. It was observed in [4, 2] that every $\lambda \in \mathbb{C}$ is an eigenvalue of A and, moreover, every $\lambda \in \mathbb{R}$ is a nonsimple eigenvalue. Theorem 3.1 shows that every $\lambda \in \mathbb{C}$ is an eigenvalue of infinite algebraic multiplicity (the geometric multiplicity of λ equals 1). Indeed, introducing as in Theorem 2.4 the operator $A_{\min} := A \cup A^*$, we see that

$$\operatorname{dom}(A) = \{y \in A_{\min}^* : \Gamma_0^+ y_+ = \Gamma_0^- y_-, \Gamma_1^+ y_+ = \Gamma_1^- y_-\},$$

where $y_+(y_-)$ is the orthoprojection of y on $L^2[0, 1]$ (resp., $L^2[-1, 0]$),

$$\Gamma_0^+ y_+ := -y'(+0), \quad \Gamma_0^- y_- := y'(-0), \quad \text{and } \Gamma_1^\pm y_\pm := y(\pm 0).$$

On the other hand, $\{\mathbb{C}, \Gamma_0^-, \Gamma_1^+\}$ is a boundary triple for $A_{\min}^+ = A_{\min} \upharpoonright L^2[0, 1]$ and $\{\mathbb{C}, \Gamma_0^-, \Gamma_1^-\}$ is a boundary triple for $A_{\min}^- = A_{\min} \upharpoonright L^2[-1, 0]$. It is easy to see that the differential expression $-\frac{d^2}{dx^2}$ is associated with both the symmetric operators A_{\min}^\pm . These operators and their boundary triples are unitarily equivalent. This means that the corresponding Weyl functions M_\pm coincide. Now Theorem 3.3 implies that any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is an eigenvalue of infinite algebraic multiplicity and therefore $\sigma(A) = \mathbb{C}$. (Actually in this case conditions (3.4) hold, so Theorem 3.1 can also be applied.) Finally, note that the functions M_\pm are meromorphic and therefore Theorem 3.3 (2)-(3) and Lemma 3.2 (ii) \Leftrightarrow (iii) imply that each point $\lambda \in \mathbb{C}$ is an eigenvalue of infinite algebraic multiplicity.

Remark 3.6. In [61], a characterization of the case $\sigma(A) = \mathbb{C}$ was given in terms of coefficients for regular operators $A = \frac{1}{r(x)} \frac{d}{dx} p(x) \frac{d}{dx}$ with Dirichlet boundary conditions. Both coefficients r and p were allowed to change sign, modifications of arguments for general regular problems were suggested also.

Arguments of Example 3.5 show that the case $\sigma(\hat{A}) = \mathbb{C}$ is exceptional in the sense of the next proposition.

Proposition 3.7. *The following statements are equivalent:*

- (i) $M_{\Sigma_+, C_+}(\lambda) = M_{\Sigma_-, C_-}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus (\operatorname{supp} d\Sigma_+ \cup \operatorname{supp} d\Sigma_-)$;
- (ii) the measures $d\Sigma_+$ and $d\Sigma_-$ coincide, and $C_+ = C_-$;
- (iii) $\sigma(\hat{A}) = \mathbb{C}$.

Moreover, if statements (i)-(iii) hold true, then every point in the set $\mathbb{C} \setminus (\sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-}))$ is an eigenvalue of \hat{A} of infinite algebraic multiplicity.

If $M_{\Sigma_+, C_+}(\cdot) \not\equiv M_{\Sigma_-, C_-}(\cdot)$, then the nonreal spectrum is the set of zeros of analytic function Φ defined in Lemma 3.2. More precisely, Theorem 3.3 shows that

$$\begin{aligned} \sigma(\hat{A}) \cap \rho(Q_{\Sigma_+} \oplus Q_{\Sigma_-}) &= \\ &= \{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) : M_{\Sigma_+, C_+}(\lambda) = M_{\Sigma_-, C_-}(\lambda)\} \subset \sigma_p(\hat{A}) \end{aligned} \quad (3.37)$$

(this statement also can be obtained from [21, Proposition 2.1]). It is easy to see that (3.37) and Theorem 3.3 yield the following description of the discrete and essential spectra (cf. [1, p. 106, Theorem 1]).

Proposition 3.8. *Assume that $M_{\Sigma_+, C_+}(\lambda_0) \neq M_{\Sigma_-, C_-}(\lambda_0)$ for certain λ_0 in the set $\rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-})$. Then:*

(i) $\sigma_{\text{ess}}(\hat{A}) = \sigma_{\text{ess}}(Q_{\Sigma_+}) \cup \sigma_{\text{ess}}(Q_{\Sigma_-}) \subset \mathbb{R};$

(ii)

$$\begin{aligned} \sigma_{\text{disc}}(\hat{A}) &= (\sigma_{\text{disc}}(Q_{\Sigma_+}) \cap \sigma_{\text{disc}}(Q_{\Sigma_-})) \cup \\ &\cup \{\lambda \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-}) : M_{\Sigma_+, C_+}(\lambda) = M_{\Sigma_-, C_-}(\lambda)\}; \end{aligned}$$

(iii) *the geometric multiplicity equals 1 for all eigenvalues of \hat{A} ;*

(iv) *if $\lambda_0 \in (\sigma_{\text{disc}}(Q_{\Sigma_+}) \cap \sigma_{\text{disc}}(Q_{\Sigma_-}))$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as a zero of the holomorphic function*

$$\frac{1}{M_{\Sigma_+, C_+}(\lambda)} - \frac{1}{M_{\Sigma_-, C_-}(\lambda)};$$

(v) *if $\lambda_0 \in \rho(Q_{\Sigma_+}) \cap \rho(Q_{\Sigma_-})$, then the algebraic multiplicity of λ_0 is equal to the multiplicity of λ_0 as zero of the holomorphic function $M_{\Sigma_+, C_+}(\lambda) - M_{\Sigma_-, C_-}(\lambda)$.*

Remark 3.9. The operator \hat{A} is definitizable if and only if the sets $\text{supp } d\Sigma_+$ and $\text{supp } d\Sigma_-$ are separated by a finite number of points (in the sense of [47, Definition 3.4]). This criterion was obtained for operators $A = (\text{sgn } x)(-d^2/dx^2 + q(x))$ in [38, 39] (see also [46, Section 2.3]) using the result of [36] and the fact that $\rho(A) \neq \emptyset$; the detailed proof was published in [47, Theorem 3.6]. The same proof is valid for the operator \hat{A} if we note that $\rho(\hat{A}) \neq \emptyset$ whenever $\text{supp } d\Sigma_+$ and $\text{supp } d\Sigma_-$ are separated by a finite number of points. Indeed, in this case $\text{supp } d\Sigma_+ \neq \text{supp } d\Sigma_-$ since $\text{supp } d\Sigma_{\pm}$ are unbounded due to the second assumption in (2.1).

3.3 Non-emptiness of resolvent set for Sturm-Liouville operators

To apply Proposition 3.8 to the J-self-adjoint Sturm-Liouville operator

$$A = \frac{\text{sgn } x}{|r(x)|} \left(-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \right)$$

introduced in Section 2.3, one has to insure that $\rho(A) \neq \emptyset$. Here we discuss briefly results of this type. We assume that $\hat{A}\{\Sigma_+, C_+, \Sigma_-, C_-\}$ is one of model operators unitarily equivalent to the operator A and that $M_{\pm}(\cdot) = M_{\Sigma_{\pm}, C_{\pm}}(\cdot)$ are the associated Weyl functions.

Sometimes it is known that the asymptotic formulae of M_+ and M_- at ∞ are different. This argument was used in [46, Proposition 2.5 (iv)] to show that $\rho(A) \neq \emptyset$ for the operator $(\operatorname{sgn} x)(-d^2/dx^2 + q(x))$. Indeed, (2.34) shows that $M_{N+}(\cdot) \not\equiv M_{N-}(\cdot)$. One can extend this result using [3, Theorem 4] in the following way: *if $p \equiv 1$ and there exist constants $r_{\pm} > 0$ such that*

$$\int_0^x (\pm r(t) - r_{\pm}) dt = o(x) \quad \text{as } x \rightarrow \pm 0,$$

then $\rho(A) \neq \emptyset$.

If $p \not\equiv 1$, one may use the standard change of variable $s = \int_0^x \frac{d\tau}{p(\tau)}$ to get back to the form with $p \equiv 1$:

Proposition 3.10. *Assume that there exist positive constants r_{\pm} such that*

$$\int_0^x r(t) dt = (r_+ + o(1)) \int_0^x \frac{dt}{p(t)} \quad \text{as } x \rightarrow +0, \quad (3.38)$$

$$\int_x^0 |r(t)| dt = (r_- + o(1)) \int_x^0 \frac{dt}{p(t)} \quad \text{as } x \rightarrow -0. \quad (3.39)$$

Then $\rho(A) \neq \emptyset$.

Another simple way to prove $\rho(A) \neq \emptyset$ uses information on the supports of spectral measures $d\Sigma_{\pm}$. In this way, it was obtained in [42, Proposition 3.1] that $\rho(A) \neq \emptyset$ if $L = \frac{1}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ is semi-bounded from below (the proof [42, p. 811] given for $p \equiv 1$ is valid in the general case). Moreover, modifying slightly the same arguments, we get the next result.

Proposition 3.11. *Assume that at least one of the symmetric operators A_{\min}^+ , A_{\min}^- (defined in Section 2.3) is semi-bounded. Then $\rho(A) \neq \emptyset$.*

Remark 3.12. (1) Proposition 3.11 has the following application to the theory of locally definitizable operators (see [35] for basic definitions): *the operator $A = \frac{\operatorname{sgn}(x)}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ introduced in Section 2.3 is locally definitizable in some open neighborhood of ∞ if and only if corresponding operator $L = \frac{1}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ is semi-bounded from below.* This is a natural generalization of [47, Theorem 3.10], where the above criterion for $r(x) = \operatorname{sgn} x$ and $p \equiv 1$ was obtained. The proof of [47, Theorem 3.10] (based on [7]) remains valid in general case if Proposition 3.11 is used instead of [46, Proposition 2.5 (iv)].

Local definitizability of Sturm-Liouville operators with the weight function r having more than one turning point was considered in [8].

(2) And vice versa, it was noticed in [47, Proposition 4.1] that local definitizability results could be used to get additional information on non-real spectrum. Namely, the above criterion of local definitizability implies that *the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of the operator $A = \frac{\operatorname{sgn}(x)}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ is bounded if the operator $L = \frac{1}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ is semi-bounded from below* (the proof is immediate from the definition of the local definitizability).

(3) Under the assumption that $\mathfrak{a}[y] = (\operatorname{sgn} x)(-y'' + qy)$ is in the limit point case in $\pm\infty$, the fact that $\rho(A) \neq \emptyset$ was noticed by M. M. Malamud and the author of this paper during the work on [45], and was published in [39, 46].

4 The absence of embedded eigenvalues and other applications

4.1 The absence of embedded eigenvalues for the case of infinite-zone potentials

Theorems 3.1 and 3.3 can be applied to prove that the Sturm-Liouville operator A has no embedded eigenvalues in the essential spectrum if some information on the spectral measures $d\Sigma_{\pm}$ is known.

We illustrate the use of this idea on operators $A = (\operatorname{sgn} x)L$, where $L = -d^2/dx^2 + q(x)$ is an operator in $L^2(\mathbb{R})$ with infinite-zone potentials q (in the sense of [55], the definition is given below). First recall that the operator $L = -d^2/dx^2 + q(x)$ with infinite zone potentials q is defined on the maximal natural domain and is self-adjoint in $L^2(\mathbb{R})$ (i.e., the differential expression is in the limit point case both at $\pm\infty$). The spectrum of L is absolutely continuous and has the zone structure, i.e.,

$$\sigma(L) = \sigma_{\text{ac}}(L) = [\mu_0^r, \mu_1^l] \cup [\mu_1^r, \mu_2^l] \cup \dots, \quad (4.1)$$

where $\{\mu_j^r\}_{j=0}^{\infty}$ and $\{\mu_j^l\}_{j=1}^{\infty}$ are sequences of real numbers such that

$$\mu_0^r < \mu_1^l < \mu_1^r < \dots < \mu_{j-1}^r < \mu_j^l < \mu_j^r < \dots, \quad (4.2)$$

and

$$\lim_{j \rightarrow \infty} \mu_j^r = \lim_{j \rightarrow \infty} \mu_j^l = +\infty.$$

μ_j^l (μ_j^r) is the left (right, resp.) endpoint of the j -th gap in the spectrum $\sigma(L)$, the "zeroth" gap is $(-\infty, \mu_0^r)$. Following [55], we briefly recall the definition of infinite-zone potential under the additional assumptions that

$$\sum_{j=1}^{\infty} \mu_j^r (\mu_j^r - \mu_j^l) < \infty, \quad \sum_{j=1}^{\infty} \frac{1}{\mu_j^l} < \infty. \quad (4.3)$$

Consider infinite sequences $\{\xi_j\}_{j=1}^{\infty}$ and $\{\epsilon_j\}_{j=1}^{\infty}$ such that $\xi_j \in [\mu_j^l, \mu_j^r]$, $\epsilon_j \in \{-1, +1\}$ for all $j \geq 1$. For every $N \in \mathbb{N}$, put

$$g_N = \prod_{j=1}^N \frac{\xi_j - \lambda}{\mu_j^l}, \quad f_N = (\lambda - \mu_0^r) \prod_{j=1}^N \frac{\lambda - \mu_j^l}{\mu_j^l} \frac{\lambda - \mu_j^r}{\mu_j^r}, \quad (4.4)$$

$$k_N(\lambda) = g_N(\lambda) \sum_{j=1}^N \frac{\epsilon_j \sqrt{-f_N(\xi_j)}}{g_N'(\xi_j)(\lambda - \xi_j)}, \quad h_N(\lambda) = \frac{f_N(\lambda) + k_N^2(\lambda)}{g_N(\lambda)}. \quad (4.5)$$

It is easy to see from (4.3) that g_N and f_N converge uniformly on every compact subset of \mathbb{C} . Denote

$$\lim_{N \rightarrow \infty} g_N(\lambda) =: g(\lambda), \quad \lim_{N \rightarrow \infty} f_N(\lambda) =: f(\lambda).$$

[55, Theorem 9.1.1] states that there exist limits

$$\lim_{N \rightarrow \infty} h_N(\lambda) =: h(\lambda), \quad \lim_{N \rightarrow \infty} k_N(\lambda) =: k(\lambda) \quad \text{for all } \lambda \in \mathbb{C}.$$

Moreover, the functions g , f , h , and k are holomorphic in \mathbb{C} .

It follows from [55, Subsection 9.1.2] that the functions

$$m_{N\pm}(\lambda) := \pm \frac{g(\lambda)}{k(\lambda) \mp i\sqrt{f(\lambda)}} \quad (4.6)$$

are the Titchmarsh-Weyl m -coefficients on \mathbb{R}_\pm (corresponding to the Neumann boundary conditions) for some Sturm-Liouville operator $L = -d^2/dx^2 + q(x)$ with a real bounded potential $q(\cdot)$. The branch $\sqrt{f(\cdot)}$ of the multifunction is chosen such that both m_\pm belong to the class (R) (see Section 2.3 for the definition).

Definition 4.1 ([55]). A real potential q is called an infinite-zone potential if the Titchmarsh-Weyl m -coefficients $m_{N\pm}$ associated with $-d^2/dx^2 + q(x)$ on \mathbb{R}_\pm admit representations (4.6).

Let q be an infinite-zone potential defined as above. B. Levitan proved that under the additional condition $\inf(\mu_{j+1}^l - \mu_j^l) > 0$, the potential q is *almost-periodical* (see [55, Chapter 11]).

The following theorem describes the structure of the spectrum of the J-self-adjoint operator $A = (\operatorname{sgn} x)L$. Note that the Titchmarsh-Weyl m -coefficients $M_{N\pm}$ for A introduced in Section 2.3 are connected with m -coefficients for L through

$$M_{N\pm}(\lambda) = \pm m_{N\pm}(\pm\lambda), \quad \lambda \in \mathbb{C} \setminus \operatorname{supp} d\Sigma_\pm \quad (\text{see e.g. [46, Section 2.2]}). \quad (4.7)$$

Theorem 4.2. *Let $L = -d^2/dx^2 + q(x)$ be a Sturm-Liouville operator with an infinite-zone potential q and let $A = (\operatorname{sgn} x)L$. Assume also that assumptions (4.3) are satisfied for the zones of the spectrum $\sigma(L)$. Then:*

- (i) $\sigma_p(A) = \sigma_{\text{disc}}(A)$, *that is all the eigenvalues of A are isolated and have finite algebraic multiplicity. Besides, all the eigenvalues and their geometric and algebraic multiplicities are given by statements (ii)-(v) of Proposition 3.8.*
- (ii) *The nonreal spectrum $\sigma(A) \setminus \mathbb{R}$ consists of a finite number of eigenvalues.*
- (iii) $\sigma_{\text{ess}}(A) = \left(\bigcup_{j=0}^{\infty} [\mu_j^r, \mu_{j+1}^l] \right) \cup \left(\bigcup_{j=0}^{\infty} [-\mu_{j+1}^l, -\mu_j^r] \right).$

The functions g , f , k , and h defined above are holomorphic in \mathbb{C} . Moreover, g and f admit the following representations

$$g(\lambda) = \prod_{j=1}^{\infty} \frac{\xi_j - \lambda}{\mu_j^l}, \quad f(\lambda) = (\lambda - \mu_0^r) \prod_{j=1}^N \frac{\lambda - \mu_j^l}{\mu_j^l} \frac{\lambda - \mu_j^r}{\mu_j^l},$$

where the infinite products converge uniformly on all compact subsets of \mathbb{C} due to assumptions (4.3) (see [55, Section 9]). It follows from (4.5) that

$$h_N(\lambda)g_N(\lambda) - k_N^2(\lambda) = f_N(\lambda) \quad \text{and} \quad h(\lambda)g(\lambda) - k^2(\lambda) = f(\lambda). \quad (4.8)$$

From this and (4.7) we get

$$M_\pm(\lambda) = \frac{g(\pm\lambda)}{k(\pm\lambda) \mp i\sqrt{f(\pm\lambda)}} = \frac{k(\pm\lambda) \pm i\sqrt{f(\pm\lambda)}}{h(\pm\lambda)}. \quad (4.9)$$

It follows from (4.8) that the zeros of the function h belongs to $(-\infty, \mu_0^r] \cup \bigcup_{j=1}^{\infty} [\mu_j^l, \mu_j^r]$. Besides, all the zeroes of h have multiplicity 1 (otherwise one of the functions $M_{N\pm}$ does not belongs to the class (R)). This implies that the spectra of the operators A_0^{\pm} defined in Section 2.3 have the following structure:

$$\sigma_{\text{ess}}(A_0^{\pm}) = \sigma_{\text{ac}}(A_0^{\pm}), \quad \sigma_{\text{ac}}(A_0^+) = -\sigma_{\text{ac}}(A_0^-) = \sigma(L) = \bigcup_{j=0}^{\infty} [\mu_j^r, \mu_{j+1}^l], \quad (4.10)$$

$$\sigma_p(A_0^{\pm}) = \sigma_{\text{disc}}(A_0^{\pm}) = \{\lambda \in \mathbb{R} \setminus \sigma(\pm L) : h(\pm\lambda) = 0, \quad k(\pm\lambda) \pm i\sqrt{f(\pm\lambda)} \neq 0\}. \quad (4.11)$$

Proof of Theorem 4.2. Since $\sigma(A) \neq \mathbb{C}$ (see Section 3.3), Proposition 3.8 (i) proves **(iii)**.

(i) We have to show that there are no eigenvalues in $\sigma_{\text{ess}}(A)$. This statement follows from the fact that $\pm \bigcup_{j=0}^{\infty} [\mu_j^r, \mu_{j+1}^l] \subset \mathfrak{A}_0(\Sigma_{N\pm})$ (see (3.1) for the definition) and Theorem 3.1 (1). Indeed, let λ_0 be in $\bigcup_{j=0}^{\infty} [\mu_j^r, \mu_{j+1}^l]$. Note that (4.10) implies that $\lambda_0 \in \sigma_{\text{ac}}(A_0^+) = \sigma_{\text{ac}}(Q_{\Sigma_{N+}})$. It follows from (4.9), (2.33) and the Stieltjes inversion formula (see e.g. [42, formula (2.9)]) that

$$\Sigma'_{N\pm}(t) = \frac{\sqrt{f(\pm t)}}{\pi h(\pm t)} \quad \text{for a.a. } t \in \pm \bigcup_{j=0}^{\infty} [\mu_j^r, \mu_{j+1}^l] \quad (4.12)$$

In particular, if $\lambda_0 \in \bigcup_{j=0}^{\infty} (\mu_j^r, \mu_{j+1}^l)$, then $\Sigma'_{N+}(t) \geq C_1$ for $|t - \lambda_0|$ small enough and a certain constant $C_1 > 0$. So

$$\int_{\mathbb{R}} |t - \lambda_0|^{-2} d\Sigma_{N+}(t) = \infty. \quad (4.13)$$

In the case when $\lambda_0 = \mu_j^r$ (or $\lambda_0 = \mu_j^l$), (4.13) also holds since $\Sigma'_+(t) \geq C_2|t - \lambda_0|^{1/2}$, $C_2 > 0$, for t in a certain left (resp., right) neighborhood of λ_0 . Arguments for $\sigma(-L) \subset \mathfrak{A}_0(\Sigma_-)$ are the same. This concludes the proof of **(i)**.

(ii) We have to prove only that $\sigma(A) \setminus \mathbb{R}$ is finite. [47, Proposition 4.1] (see also Remark 3.12 (2)) implies that

$$\sigma(A) \setminus \mathbb{R} \quad \text{is a bounded set.} \quad (4.14)$$

Proposition 3.8 (ii) states that points of $\sigma(A) \setminus \mathbb{R}$ are zeros of the holomorphic in $\mathbb{C} \setminus (\text{supp } d\Sigma_{N+} \cup \text{supp } d\Sigma_{N-})$ function $M_{N+}(\lambda) - M_{N-}(\lambda)$, and therefore, are roots of each of the following equations (each subsequent equation is a modification of the previous one)

$$\frac{g(\lambda)}{k(\lambda) - i\sqrt{f(\lambda)}} = \frac{g(-\lambda)}{k(-\lambda) + i\sqrt{f(-\lambda)}},$$

$$\begin{aligned} (g(\lambda)k(-\lambda) - g(-\lambda)k(\lambda))^2 = \\ = -g^2(\lambda)f(-\lambda) - g^2(-\lambda)f(\lambda) - 2g(\lambda)g(-\lambda)\sqrt{f(-\lambda)}\sqrt{f(\lambda)}, \end{aligned}$$

$$\begin{aligned} \left([g(\lambda)k(-\lambda) - g(-\lambda)k(\lambda)]^2 + g^2(\lambda)f(-\lambda) + g^2(-\lambda)f(\lambda) \right)^2 - \\ - 4g^2(\lambda)g^2(-\lambda)f(-\lambda)f(\lambda) = 0. \end{aligned}$$

The entire function in the left side of the last equation is not identically zero since it is positive in the points of the set

$$\{\lambda \in (\sigma(L) \setminus \sigma(-L)) \cup (\sigma(-L) \setminus \sigma(L)) : g(\lambda)g(-\lambda) \neq 0\}$$

(this set is nonempty due to (4.1), (4.6), and the fact that $M_{N\pm}(\cdot) \not\equiv 0$). Therefore,

$$\begin{aligned} M_{N+} - M_{N-} \quad & \text{has a finite number of zeros} \\ & \text{in any bounded subset of } \mathbb{C} \setminus (\text{supp } d\Sigma_{N+} \cup \text{supp } d\Sigma_{N-}) \quad (\supset \mathbb{C} \setminus \mathbb{R}). \end{aligned} \quad (4.15)$$

Combining this and (4.14), we see that $\sigma(A) \setminus \mathbb{R}$ is finite. \square

Note that (4.15) implies that each gap of the essential spectrum $\sigma_{\text{ess}}(A)$ has at most finite number of eigenvalues.

4.2 Other applications

A part of this paper was obtained in author's candidate thesis [38], was announced together with some applications in the short communication [39], and was a base of several author's conference talks in 2004-2005. Namely, in [38, 39], the operator $\hat{A}\{\Sigma_+, C_+, \Sigma_-, C_-\}$ was introduced as a functional model for the operator $(\text{sgn } x)(-d^2/dx^2 + q)$ (see Theorem 2.6) and the description of eigenvalues under conditions (3.4) was given (see Theorem 3.1). These results were used in [46, 42]. The idea of the functional model originated from [41, 45], where a representation of the operator $(\text{sgn } x)(-d^2/dx^2 + q)$ as an extension of a direct sum of two symmetric Sturm-Liouville operators was used essentially (in a less explicit form the same idea appeared earlier in [17, 28]).

The absence of embedded eigenvalues in the essential spectrum of the operator $(\text{sgn } x)(-d^2/dx^2 + q)$ with a finite-zone potential q was proved in [46, Theorem 7.1 (2)] via Theorem 3.1. This proof is adopted in part (i) of Theorem 4.2 for the infinite-zone case.

Theorem 3.1 helps to find algebraic multiplicity of embedded eigenvalues. This was used in a paper of A. Kostenko and the author (see [42], Proposition 2.2, Theorems 6.1 (ii) and 6.4 (ii)) to prove simplicity of the eigenvalue $\lambda = 0$ for two operators of type $(\text{sgn } x)(-d^2/dx^2 + q)$. This fact and necessary conditions for regularity of critical points (see [42, Theorem 3.9]) allowed us to show that 0 is a singular critical point for the considered operators (see [42, Remark 6.3, Theorem 6.4 (iii)] and also Section 5 of the present paper).

The fact that 0 may be a non-semi-simple eigenvalue (i.e., $\ker A^2 \neq \ker A$) is essential for the theory of "two-way" diffusion equations. In the simplest case, such equations lead to spectral analysis of J -nonnegative operators that take the form of the operator A introduced in Section 2.3. If 0 and ∞ are not singular critical points of A than the algebraic multiplicity of 0 affects proper settings of boundary value problems for the corresponding diffusion equation (see [6, 5, 32]). If 0 is a singular critical point of A , as in the examples constructed in [42, 44] and Section 5, the existence and uniqueness theory for corresponding diffusion equations is not well-understood (see [48, Section 1], [14, 59, 40]).

Remark 4.3. For periodic potentials and certain classes of decaying potentials, asymptotic behavior of solutions of $-y'' + qy = \lambda y$ is well-known and yields the absence of eigenvalues in some parts of the spectrum of the operator $A = (\text{sgn } x)L$ (where $L = -d^2/dx^2 + q$ is a self-adjoint operator in $L^2(\mathbb{R})$).

For example, the assumption $\int_{-\infty}^{+\infty} (1+|x|)|q(x)|dx < \infty$ yields $\sigma_p(A) \cap \mathbb{R} = \emptyset$ (see e.g. [58, Lemma 3.1.1 and formula (3.2.4)]). This fact was used essentially in [44, Section 4] to prove that, for this class of potentials, A is similar to self-adjoint operator exactly when $L \geq 0$. For periodic potentials, $\sigma_{\text{ess}}(A) \cap \sigma_p(A) = \emptyset$ follows from [63, Section 21]. For $q \in L^1(\mathbb{R})$, the fact that the only possible real eigenvalue is 0 (i.e., $\sigma_p(A) \cap \mathbb{R} \subset \{0\}$) follows immediately from [63, Section 5.7] or from [13, Problem IX.4].

5 Remarks on indefinite Sturm-Liouville operators with the singular critical point 0

In this section, we provide an alternative approach to examples of J -self-adjoint Sturm-Liouville operators with the singular critical point 0 constructed in [42, Sections 5 and 6] and [44, Section 5].

As before, let H be a Hilbert space with a scalar product $(\cdot, \cdot)_H$. Suppose that $H = H_+ \oplus H_-$, where H_+ and H_- are (closed) subspaces of H . Denote by P_{\pm} the orthogonal projections from H onto H_{\pm} . Let $J = P_+ - P_-$ and $[\cdot, \cdot] := (J\cdot, \cdot)_H$. Then the pair $K = (H, [\cdot, \cdot])$ is called a *Krein space*. The operator J is called a fundamental symmetry (or a signature operator) in the Krein space K . Basic facts on the theory of Krein spaces and the theory of J -self-adjoint definitizable operators can be found e.g. in [54]. An account on (non-differential) operators with a finite singular critical point can be found in [15]. The following proposition is a simple consequence of [54, Theorem II.5.7].

Proposition 5.1. *Assume that a J -self-adjoint definitizable operator B has a simple eigenvalue $\lambda_0 \in \mathbb{R}$ and that a corresponding eigenvector $f_0 \in \ker(B - \lambda_0) \setminus \{0\}$ is neutral (i.e., $[f_0, f_0] = 0$). Then λ_0 is a singular critical point of B .*

Proof. Assume that λ_0 is not a singular critical point of B . Then there exist a J -orthogonal decomposition into a direct sum of two closed subspaces $H = H_0 \dot{+} H_1$ that reduces the operator B and such that H_0 is a root subspace corresponding to λ_0 and $\lambda_0 \notin \sigma_p(B \upharpoonright H_1)$ (this follows from [54], Proposition II.5.1, Proposition II.5.6, Theorem II.5.7, and the decomposition (II.2.10)). Since the eigenvalue λ_0 is simple, we have $H_0 = \{cf_0, c \in \mathbb{C}\}$. Since f_0 is neutral, we see that $[f_0, f] = 0$ for all f in H . So $\{H, [\cdot, \cdot]\}$ is not a Krein space, a contradiction. \square

It is not difficult to see that the operators considered in [42, Sections 5 and 6] and [44, Section 5] satisfy conditions of Proposition 5.1, and so Proposition 5.1 can be used in these two papers instead of [42, Theorem 3.4]. This does not makes the proofs much simpler, but the results become clearer from the Krein space point of view.

Remark 5.2. The necessary similarity condition given in [42, Theorem 3.4] is of independent interest since it provides a criterion of similarity to a self-adjoint operator for operators $(\text{sgn } x)(-d^2/dx^2 + q)$ with finite-zone potentials (see [42, Remark 3.7]). And it is unknown *whether the condition of [42, Theorem 3.4] provides a criterion of similarity to a self-adjoint operator for the general operator $\frac{\text{sgn } x}{|r|}(-\frac{d}{dx}p\frac{d}{dx} + q)$ with one turning point introduced in Section 2.3.*

Using Proposition 5.1, a large class of operators with the singular critical point 0 similar to that of [42, 44] can be constructed. In the next theorem, we characterize the case described in Proposition 5.1 among the operators $\mathcal{A}_r := -\frac{\text{sgn } x}{|r(x)|}\frac{d^2}{dx^2}$ that have the limit point case both at $\pm\infty$ (and act in

$L^2(\mathbb{R}; |r(x)|dx)$). So we assume that

$$r \in L^1_{\text{loc}}(\mathbb{R}), \quad r(x) = (\text{sgn } x)|r(x)|, \quad (5.1)$$

$$\int_{\mathbb{R}_{\pm}} x^2 |r(x)| dx = \infty, \quad (5.2)$$

and the operator \mathcal{A}_r is defined on its maximal domain. Condition (5.2) is equivalent to J -self-adjointness of \mathcal{A}_r with $J : f(x) \mapsto (\text{sgn } x)f(x)$. Obviously, \mathcal{A}_r is J -non-negative and definitizable (see [16] and also Proposition 3.11).

Theorem 5.3. *Let assumptions (5.1), (5.2) be satisfied. Then:*

(1) *Two following statements are equivalent:*

(r1) \mathcal{A}_r has a simple eigenvalue at 0 and $[f_0, f_0] = 0$ for $f_0 \in \ker \mathcal{A}_r \setminus \{0\}$.

(r2) $r \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} r(x) dx = 0$, and

$$\int_{\mathbb{R}} (y_1(x))^2 |r(x)| dx = +\infty, \quad \text{where } y_1(x) := \int_0^x \int_s^{+\infty} r(t) dt ds. \quad (5.3)$$

(2) *If (r2) is satisfied, then 0 is a singular critical point of \mathcal{A}_r .*

Proof. We need to prove only (1). Let us note that $r \in L^1(\mathbb{R})$ is equivalent to $0 \in \sigma_p(\mathcal{A}_r)$. If the latter holds, then $f_0(x) \equiv 1$ is an eigenfunction (unique up to multiplication by a constant), and $[f_0, f_0] = 0$ is equivalent to $\int_{\mathbb{R}} r(x) dx = 0$.

Assume that the eigenvalue 0 is not simple. Then there is a generalized eigenfunction of first order $y \in L^2(\mathbb{R}, |r(x)|dx)$, which is a solution of $\mathcal{A}_r y = f_0$. It is easy to see that the derivative of y has the form $y' = y'_1 + C_1$, where $C_1 \in \mathbb{C}$ is a constant and y_1 is defined by (5.3). Condition $y \in L^2(\mathbb{R}, |r(x)|dx)$ implies that $C_1 = 0$ (otherwise $y'(x) \rightarrow C_1 \neq 0$ as $x \rightarrow \pm\infty$ and, therefore, $y \notin L^2(\mathbb{R}; |r(x)|dx)$ due to (5.2)). This shows that y_1 is the only possible generalized eigenvector, and (5.3) ensures that $y_1 \notin L^2(\mathbb{R}, |r(x)|dx)$. Thus, (5.3) is equivalent to the fact that 0 is a simple eigenvalue (under the assumptions $r \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} r(x) dx = 0$). \square

In the following corollary $f(x) \asymp g(x)$ as $x \rightarrow +\infty$ ($x \rightarrow -\infty$) means that for $X > 0$ large enough both $\frac{f}{g}$ and $\frac{g}{f}$ are bounded on $(X, +\infty)$ (resp., $(-\infty, -X)$).

Corollary 5.4. *Assume that condition (5.1) is satisfied and $r(x) \asymp \pm|x|^\alpha$ as $x \rightarrow \pm\infty$. If $\alpha \in [-5/3, -1)$ and $\int_{\mathbb{R}} r(x) dx = 0$, then the operator \mathcal{A}_r has a singular critical point at 0.*

This includes [42, Theorem 5.1 (ii)-(iV)], also it is interesting to compare this result with the sufficient condition on regularity of 0 given in [51], [44, Theorem 1.3], and the discussions in [44, Section 5.2] and [43, Section 5].

Remark 5.5. Theorem 5.3 can be easily generalized to weight functions r with many turning points under the assumption that

$$r(x) \text{ is of constant sign a.e. on } (-\infty, -X) \text{ and } (X, +\infty) \quad (5.4)$$

for certain $X > 0$ large enough. Indeed, assume additionally $r \in L^1_{\text{loc}}(\mathbb{R})$, $r \neq 0$ a.e. on \mathbb{R} , and (5.2). [16, Proposition 2.5] implies that the maximal operator $\mathcal{A}_r := -\frac{1}{r} \frac{d^2}{dx^2}$ is definitizable. Now it is easy to see that statements (1) and (2) of Theorem 5.3 are valid with the same proof for $\mathcal{A}_r = -\frac{1}{r} \frac{d^2}{dx^2}$.

6 Discussion

From another point of view algebraic multiplicities of eigenvalues of definitizable operators was considered in [54, Proposition II.2.1] and [16, Section 1.3] in terms of definitizing polynomials. For operators $\frac{\operatorname{sgn} x}{|r(x)|}(-\frac{d}{dx}p\frac{d}{dx} + q)$, Theorem 3.3 solves the same problem in terms of Titchmarsh-Weyl m -coefficients. Combining both approaches, it is possible to get quite precise results both on eigenvalues and on definitizing polynomials. Such analysis was done in [44, Section 4.3] for operators $(\operatorname{sgn} x)(-d^2/dx^2 + q)$ with potentials $q \in L^1(\mathbb{R}; (1 + |x|)dx)$; in particular, the minimal definitizing polynomial was described in terms of Titchmarsh-Weyl m -coefficients (recently, [52] was used in [9] to extended some results of [44, Section 4.3] on a slightly more general class of potentials).

A. Kostenko and later B. Ćurgus informed the author that Theorem 3.1 and [42, Section 6.1] are in disagreement with one of the statements of [16, p. 39, 1st paragraph].

Namely, [16, Section 1.3] is concerned with J -self-adjoint operators A (in a Krein space $K = (H, [\cdot, \cdot])$) such that the form $[A\cdot, \cdot]$ has a finite number k_A of negative squares. Such operators are sometimes called quasi- J -nonnegative. It is assumed also that $\rho(A) \neq \emptyset$. Here, as before, H is a Hilbert space, J is a fundamental symmetry, and $[\cdot, \cdot] := (J\cdot, \cdot)_H$.

According to [16, p. 39, 1st paragraph] (the author changes slightly the appearance):

- (p1) an operator A of the type mentioned above has a definitizing polynomial p_A of the form $p_A(z) = zq_A(z)\overline{q_A(\bar{z})}$, where the polynomial q_A can be chosen monic and of minimal degree. Under these assumptions, q_A is unique and its degree is less than or equal to k_A .
- (p2) A real number $\lambda \neq 0$ is a zero of q_A if and only if it is an eigenvalue of A such that $\lambda[f, f] \leq 0$ for some corresponding eigenvector f .
- (p3) $q_A(0) = 0$ implies that 0 is an eigenvalue of A and one of corresponding Jordan chains is of length ≥ 2 .

Note also that in the settings of [16, p. 39, 1st paragraph], q_A is of minimal degree, but it easy to see the definitizing polynomial p_A may be not a definitizing polynomial of minimal degree.

From the author's point of view, two following statements in [16, Section 1.3] are incorrect: assertion (p3) and the equality $\dim \mathcal{L}_0 = k_A + k_A^0$ in [16, Proposition 1.5]. Statement (p3) was given as a simple consequence of considerations in [54]. The proof of [16, Proposition 1.5] has an unclear point, which is discussed below. As far as the author understand, all other results of [16] (as well as results obtained in [44, Section 4.3]) do not depend of these two statements.

Let us explain points of contradiction in more details. In [42, Section 6.1], the operator A , defined by

$$A := JL, \quad (Jf)(x) = (\operatorname{sgn} x)f(x), \quad (Ly)(x) = -y''(x) + 6\frac{x^4 - 6|x|}{(|x|^3 + 3)^2}y(x), \quad (6.1)$$

is considered in $L^2(\mathbb{R})$. The operators L and A are defined on their maximal domains, L is self-adjoint, and A is J -self-adjoint. We will use here notations of Section 2.3. In particular, $\mathfrak{l}[\cdot]$ is the differential expression of the operator L .

We will need the following properties of the operator A .

Proposition 6.1 (cf. Section 6.1 in [42]). *Let A be the J -self-adjoint operator defined by (6.1). Then:*

- (i) A is a quasi- J -nonnegative operator, with $k_A = 1$, i.e., the sesquilinear form $[A\cdot, \cdot]$ has one negative square;
- (ii) $\sigma(A) \subset \mathbb{R}$;
- (iii) $\sigma_p(A) = \{0\}$;
- (iv) 0 is a simple eigenvalue of A .

Statements (ii) and (iv) of Proposition 6.1 were proved in [42, Theorem 6.1]. Statement (i) was given in [42, Remark 6.3] with a very shortened proof. For the sake of completeness, we give below the proofs of statements (i) and (iii).

Proof. Statement (i) follows from [16, Remark 1.2] and the fact that the negative part of the spectrum of the operator L consists of one simple eigenvalue at $\lambda_0 = -1$.

Let us prove that $\sigma(L) \cap (-\infty, 0) = \{-1\}$. Consider the operators L_0^\pm associated with the differential expression $\mathfrak{l}[\cdot]$ and the Neumann problems $y'(\pm 0) = 0$ on \mathbb{R}_\pm , and let $m_{N\pm}$ be the corresponding Titchmarsh-Weyl m -coefficients, see e.g. [44, formula (2.7)].

It follows from [42, Lemma 6.2] that both the Titchmarsh-Weyl m -coefficient $m_{N\pm}$ are equal to the function m_0 defined by

$$m_0(\lambda) = \frac{\lambda}{1 + \lambda(-\lambda)^{1/2}}, \quad \lambda \in \mathbb{C} \setminus (\{-1\} \cup [0, +\infty)), \quad (6.2)$$

where $z^{1/2}$ denotes the branch of the complex root with a cut along the negative semi-axis \mathbb{R}_- such that $(-1 + i0)^{1/2} = i$.

$\lambda_0 = -1$ is a pole of m_0 , and therefore, an eigenvalue of both the operators L_0^+ , L_0^- , and in turn an eigenvalue of L .

The support of the spectral measure of m_0 is equal to $\{-1\} \cup [0, +\infty)$, see part (ii) of the proof of [42, Theorem 6.1]. It is easy to see from the standard definition of Titchmarsh-Weyl m -coefficients (or from [21, Proposition 2.1]) that $\lambda \in (-\infty, -1) \cup (-1, 0)$ belongs to the spectrum of L if and only if $m_{N+}(\lambda) + m_{N-}(\lambda) = 0$. But [42, formula (6.2)] implies that $m_{N+}(\lambda) + m_{N-}(\lambda) = 2m_0(\lambda) \neq 0$ for all $\lambda \in (-\infty, -1) \cup (-1, 0)$.

Thus, the eigenvalue $\lambda_0 = -1$ is the only point of the spectrum of L in $(-\infty, 0)$. $\lambda_0 = -1$ is a simple eigenvalue since $\mathfrak{l}[\cdot]$ is in the limit point case at $\pm\infty$ (see [65, Theorem 5.3]).

(iii) It is proved in [42, Theorem 6.1] that 0 is an eigenvalue of A and that $\sigma(A) \subset \mathbb{R}$. Here we have to show that A has no eigenvalues in $\mathbb{R} \setminus \{0\}$.

The proof of [42, Theorem 6.1 (ii)] states that the Titchmarsh-Weyl m -coefficient M_{N+} is equal to m_0 , that its spectral measure Σ_{N+} is absolutely continuous on intervals $[0, X]$, $X > 0$, and that for $t > 0$ we have

$$\Sigma'_{N+}(t) := \frac{t^{5/2}}{\pi(1 + t^3)}.$$

Combining this with Theorem 3.1 (1), we see that $(0, +\infty) \subset \mathfrak{A}_0(\Sigma_{N+})$, and therefore $\sigma_p(A) \cap (0, +\infty) = \emptyset$. Since the potential of L is even, we see that $\sigma_p(A) \cap (-\infty, 0) = \emptyset$. This concludes the proof. \square

Remark 6.2. Actually, $\sigma(A) = \mathbb{R}$. This follows from Proposition 3.8 (i) and the fact that $M_{N\pm}(\cdot) = \pm m_{N\pm}(\pm \cdot)$. The fact that A has no eigenvalues in $\mathbb{R} \setminus \{0\}$ can also be easily obtained from [63, Section 5.7] or from [13, Problem IX.4].

Combining Proposition 6.1 with **(p1)** and **(p2)**, we will show that $q_A(z) = z$, and that $q_A(z) = z$ contradicts **(p3)**. Indeed, since L is not nonnegative, the polynomial z is not a definitizing polynomial of the operator A . So $p_A(z) \not\equiv z$, and therefore q_A is nontrivial. q_A has the degree equal to $k_A = 1$ due to **(p1)**. Since the polynomial q_A is of minimal degree, Proposition 6.1 (ii) implies that q_A has no non-real zeros, see [54, p.11, the second paragraph] or [16, p. 38, the last paragraph]. (Note also that in our case p_A is a definitizing polynomial of minimal degree since 0 is a critical point of A .) By Proposition 6.1 (iii), A has no eigenvalues in $\mathbb{R} \setminus \{0\}$. Therefore, statement **(p2)** implies that q_A has no zeros in $\mathbb{R} \setminus \{0\}$. Summarizing, we see that $q_A(z) = z$ and $p_A(z) = z^3$. Proposition 6.1 (iv) states that 0 is a simple eigenvalue. This fact contradicts **(p3)**.

The equality $\dim \mathcal{L}_0 = k_A + k_A^0$ from [16, Proposition 1.5] is not valid for the operator A defined by (6.1).

Namely, [16, Proposition 1.5] states that there exists an invariant under A subspace \mathcal{L}_0 of dimension $\dim \mathcal{L}_0 = k_A + k_A^0$, where k_A^0 is the dimension of the isotropic part of the root subspace $\mathcal{S}_0(A)$ with respect to the sesquilinear form $[A\cdot, \cdot]$. For the operator A defined by (6.1), statements (i) and (iv) of Proposition 6.1 imply that $k_A = 1$ and $k_A^0 = 1$, respectively. So \mathcal{L}_0 is a two-dimensional invariant subspace of A . All the root subspaces of the restriction $A \upharpoonright \mathcal{L}_0$ are root subspaces of A , and therefore Proposition 6.1 (iii)-(iv) implies $\dim \mathcal{L}_0 \leq 1$. This contradicts $\dim \mathcal{L}_0 = k_A + k_A^0 = 2$.

Remark 6.3. From the author's point of view, the statement 'the inner product $[A\cdot, \cdot]$ has k_A'' negative squares on $\mathcal{S}_0(A)$ ' in the proof of [16, Proposition 1.5] is not valid for the operator A defined by (6.1), since in this case $\mathcal{S}_0(A) = \ker A$, but $k_A'' = 1$.

A Appendix: Boundary triplets for symmetric operators

In this section we recall necessary definitions and facts from the theory of boundary triplets and abstract Weyl functions following [49, 31, 21, 22].

Let $H, \mathcal{H}, \mathcal{H}_1$, and \mathcal{H}_2 be complex Hilbert spaces. By $[\mathcal{H}_1, \mathcal{H}_2]$ we denote the set of bounded linear operators acting from the space \mathcal{H}_1 to the space \mathcal{H}_2 and defined on all the space \mathcal{H}_1 . If $\mathcal{H}_1 = \mathcal{H}_2$, we write $[\mathcal{H}_1]$ instead of $[\mathcal{H}_1, \mathcal{H}_1]$.

Let S be a closed densely defined symmetric operator in H with equal deficiency indices $n_+(S) = n_-(S) = n$ (by definition, $n_{\pm}(S) := \dim \mathfrak{N}_{\pm i}(S)$, where $\mathfrak{N}_{\lambda}(S) := \ker(S^* - \lambda I)$).

Definition A.1. A triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of an auxiliary Hilbert space \mathcal{H} and linear mappings $\Gamma_j : \text{dom}(S^*) \longrightarrow \mathcal{H}$, ($j = 0, 1$), is called a *boundary triplet for S^** if the following two conditions are satisfied:

- (i) $(S^*f, g)_H - (f, S^*g)_H = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}}, \quad f, g \in \text{dom}(S^*);$
- (ii) the linear mapping $\Gamma = \{\Gamma_0 f, \Gamma_1 f\} : \text{dom}(S^*) \longrightarrow \mathcal{H} \oplus \mathcal{H}$ is surjective.

In the rest of this section we assume that the Hilbert space H is separable. Then the existence of a boundary triplet for S^* is equivalent to $n_+(S) = n_-(S)$.

The mappings Γ_0 and Γ_1 naturally induce two extensions S_0 and S_1 of S given by

$$S_j := S^* \upharpoonright \operatorname{dom}(S_j), \quad \operatorname{dom}(S_j) = \ker \Gamma_j, \quad j = 0, 1.$$

It turns out that S_0 and S_1 are self-adjoint operators in H , $S_j^* = S_j$, $j = 0, 1$.

The γ -field of the operator S corresponding to the boundary triplet Π is the operator function $\gamma(\cdot) : \rho(S_0) \rightarrow [\mathcal{H}, \mathfrak{N}_\lambda(S)]$ defined by $\gamma(\lambda) := (\Gamma_0 \upharpoonright \mathfrak{N}_\lambda(S))^{-1}$. The function γ is well-defined and holomorphic on $\rho(S_0)$.

Definition A.2 ([21, 22]). Let $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ be a boundary triplet for the operator S^* . The operator-valued function $M(\cdot) : \rho(S_0) \rightarrow [\mathcal{H}]$ defined by

$$M(\lambda) := \Gamma_1 \gamma(\lambda), \quad \lambda \in \rho(S_0),$$

is called *the Weyl function* of S corresponding to the boundary triplet Π .

Note that the Weyl function M is holomorphic on $\rho(S_0)$ and is an (operator-valued) (R) -function obeying $0 \in \rho(\operatorname{Im}(M(i)))$.

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